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# INFINITESIMAL CALCULUS

Longmans' Modern Mathematical Series

# INFINITESIMAL CALCULUS

*SECTION I*

BY

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*WITH DIAGRAMS*

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# PREFACE

THIS book is divided into two sections: the first deals with those parts of the Infinitesimal Calculus which have been recently introduced into the syllabus of some examinations for higher school certificates, while the two sections taken together correspond fairly closely to the curriculum of students reading for the first part of an honours course in mathematics or for the ordinary degree in arts, science or engineering.

Believing that there is no royal road which leads smoothly and directly to the Infinitesimal Calculus, the author has made no attempt to evade all the difficulties which at the outset face the student in this subject. The road has, however, been laid in the first section so as to pass through those domains of number and function with which the student is probably already acquainted, while the functions which are likely to be unfamiliar to him have been reserved for the second section.

To assist the student in mastering the fundamental conception of a differential coefficient, two ideas which are usually reserved for books of a more advanced character have been introduced at the beginning and used throughout the book, namely, range and sequence, and the ordinary symbolism in connection with them has been varied. The symbols to express open and closed ranges have, however, been changed so slightly that the alteration amounts to little more than a typographical modification; but a more important change has been made in the arrow notation, which is now so generally used to express the limits of a sequence. The author is indebted to his former colleague, Dr. James Mercer, for the beautiful suggestion of the arrow with a single barb, either upper or lower. This notation has been successfully tested in class work, but has not previously appeared in print. Teachers who have recognised the great advantages which have resulted from the

introduction of the fully barbed arrow may perhaps be willing to try the experiment of arrows with a single barb.

No attempt has been made in the first section to deal with the definite integral, nor has the usual notation for the indefinite integral been introduced until a comparatively advanced stage. It need hardly be said that it is no part of the author's plan to exclude such an important and universally adopted symbol; his plea for the postponement of its use is based upon the impossibility of justifying the symbol $\int\varphi(x)dx$ as a representation of inverse differentiation until the nature of a definite integral has been explained. Something also may be gained by introducing students at an early stage to a differential equation, even though it is of the simple type

$$\frac{dy}{dx} = \varphi(x)$$

The book is not written for any particular group of students; it is designed for those who wish to use the Infinitesimal Calculus as an instrument in the attainment of further knowledge. It is therefore essentially a book of practical mathematics. With this end in view, fundamental ideas are explained at great length; for the easiest and quickest way to master this subject is to acquire a firm grasp of the conceptions upon which it is based. The student is advised to return again and again to the earlier chapters; it is only gradually that the matter contained in them can be assimilated, and the knowledge which the student acquires as he progresses will sometimes furnish the key to open some doors that may be closed to him at a first attempt. The Infinitesimal Calculus cannot be learnt either by memory or by mimicry—it needs judgment and reflection; but if studied in the right spirit it may, by strengthening these qualities of mind, fulfil one of the most valuable aims of education.

Historical notes and references have not been given. The student who is interested in this side of the subject should turn to Dr. Whitehead's *Introduction to Mathematics*, and to Mr. Jourdain's *Nature of Mathematics*; these valuable books are readily accessible, and will naturally conduct the student to Mr. Rouse Ball's more systematic writings on the history of the subject. In these various books he will find the natural complement and the fitting reward of the serious work which the study of a text-book implies.

The author acknowledges most gratefully the help of many friends. Dr. Macaulay, an editor of the series in which the book appears, has given practical assistance in reading the manuscript and in following the book through the press; he has also given a generous encouragement, which has been as invaluable as his criticism. Dr. Proudman has read the earlier chapters in manuscript more than once, and proofs of the whole; there are many parts of the book which have assumed their final shape under his skilful direction. Mr. R. Hargreaves and Mr. R. O. Street have read the proofs, and a deep debt is owed to them for many valuable criticisms and useful suggestions.

The exercises are numerous, as they must be in a book on this subject. They afford examples for the student both on his first reading and also when he returns to the subject and goes over it a second or a third time. The author will welcome corrections of errors. In cases where it seems desirable that the attention of the solver should not be distracted by the form of the answers, these have been placed at the end of the book.

F. S. CAREY.

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## *SECTION I*

# CHAPTER I

## NUMBER, FUNCTION, GRAPH

**1. Numbers.**

The numbers which are used in Arithmetic and Algebra are often qualified by adjectives ; thus, we speak of positive and of negative numbers, of integral and of fractional numbers. By each of such qualifications the existence of a *class* of numbers is suggested. For if we say that 2 is an integer, we imply (i) that there are other integers, such as 3, 4, 5, ... , and (ii) that there are numbers, such as $\frac{1}{2}$, $\frac{2}{3}$, ... , which are not integral.

Numbers are divided first of all into positive and negative numbers, zero not belonging to either class. To every positive number there corresponds a negative number such that the addition of these two numbers is zero, for example,

$$2 + (-2) = 0$$

In the further classification of numbers which we make here, we shall consider only the class of positive numbers, remembering that to each positive number there corresponds a negative number.

Integers claim our attention first. They may be written in the following natural order

$$1,\ 2,\ 3,\ 4, \ldots$$

Such a set is the type of a sequence, or progression, of numbers, the distinguishing property of a *sequence* of numbers being that it is always possible to write down the number which occupies the $n$th place. An example of such a sequence with which the student is familiar is given by

$$10,\ 8,\ 6,\ 4, \ldots$$

in which the $n$th place is occupied by the number $12 - 2n$ ; this sequence is an arithmetical progression. Other familiar instances of sequences of integers are

$$1,\quad 2,\quad 4,\quad 8, \ldots\ 2^{n-1}, \ldots$$

$$2,\quad 6,\quad 12,\quad 20, \ldots\ n(n+1), \ldots$$

$$1,\ -1,\quad 1,\ -1, \ldots\ (-1)^{n-1}, \ldots$$

After integers the class of fractions has to be taken. Fractions are divided into various sub-classes; we have reduced and non-reduced fractions; some fractions also are proper, while others are spoken of as improper. But, whatever sub-classes are introduced, all fractions are included in the sequence

$$\tfrac{1}{2};\ \tfrac{1}{3}, \tfrac{2}{2};\ \tfrac{1}{4}, \tfrac{2}{3}, \tfrac{3}{2};\ \tfrac{1}{5}, \tfrac{2}{4}, \tfrac{3}{3}, \tfrac{4}{2};\ \tfrac{1}{6}, \tfrac{2}{5}, \tfrac{3}{4}, \tfrac{4}{3}, \tfrac{5}{2};\ldots$$

In this sequence the reader will notice that $\frac{1}{2}$ and $2/4$ both occur. It is true that these are equal, but there is a sense in which they may be distinguished; for $\frac{1}{2}$ implies the division of the unit into two parts, whereas $2/4$ implies that it is divided into four parts. Again, the inclusion of such numbers as $3/3$ and $4/2$ may seem to outstrip the ordinary interpretation of fractions. Those who feel doubt about the inclusion of such numbers in the above catalogue may strike them out and be assured that the reduced list contains all the numbers which they regard as fractions. The sequence is divided into blocks by semi-colons, the successive blocks containing every fraction the sum of whose numerator and denominator is equal to 3, 4, 5, 6, 7, ... Thus, in the 20th block the fractions would be

$$\tfrac{1}{21}, \tfrac{2}{20}, \tfrac{3}{19}, \ldots \tfrac{20}{2}$$

In the sequence it is possible to place any given fraction; thus, if $19/21$ is considered, the sum of its numerator and denominator is 40; this fact places it in the 38th block; its actual place in the complete sequence is found to be 722. The fractions in each block of the sequence are arranged in order of magnitude; in the sequence, as a whole, this is not the case. It is, in fact, clearly impossible to arrange fractions in order of magnitude: for, between any two given positive fractions, $a/b$ and $c/d$, the fraction $(a + c)/(b + d)$ lies.

Positive integers and fractions can be derived from unity by a finite number of two of the fundamental operations of Arithmetic, addition and division. Thus we have as illustrations

$$5 = 1 + 1 + 1 + 1 + 1$$
$$\frac{3}{4} = \frac{1 + 1 + 1}{1 + 1 + 1 + 1}$$

But there are numbers which cannot be obtained by a *finite* number of these operations; some such numbers are familiar to the reader, *e.g.*,

$$\sqrt{2},\quad \sqrt{3},\quad \sqrt[3]{5},\quad \pi,\quad e,\quad \log_{10}2, \ldots$$

These are called *irrational* or *incommensurable* numbers; they will be discussed below.

**2. Functions.**

The simplest illustration of a function is provided by a mathematical table, which, however it is arranged, consists essentially

of two columns, or rows, of numbers placed so that the numbers which are opposite to each other correspond. Ordinary mathematical tables contain information about squares, square roots, cubes, cube roots, logarithms, antilogarithms, sines, cosines, tangents, log sines, log cosines, and log tangents. In the case of each of these tables there is a process which allows us to verify that the number on the right hand corresponds to the number on the left when the table is arranged in column ; moreover, the same process which gives the numbers in the table allows of its extension to include pairs of corresponding numbers which are not given in the table. If we look first at a simple case, the table of squares, and arrange the table horizontally, we have as an abbreviated specimen of a table

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|
| $x^2$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 | 16 |

Here $x$ and $x^2$ are *variables*, that is, symbols which stand for any number from a given class. This table might be extended at any point ; thus between 1 and 2 we might insert nine numbers by taking $x$ to every tenth,

| $x$ | 1·1 | 1·2 | 1·3 | 1·4 | 1·5 | 1·6 | 1·7 | 1·8 | 1·9 |
|---|---|---|---|---|---|---|---|---|---|
| $x^2$ | 1·21 | 1·44 | 1·69 | 1·96 | 2·25 | 2·56 | 2·89 | 3·24 | 3·61 |

In general, *a correspondence between two classes of numbers in which to each number of the first class corresponds a number of the second class is called a functional relation.* Again, the variable which corresponds to numbers in the first class is called the *independent variable,* and that which corresponds to those in the second class is the *dependent variable.* In this way we may say that there is a functional relation between the independent and dependent variables, or, as it is more usually stated, the dependent variable is a *function* of the independent variable.

In the first section of this book the reader's attention is specially directed to single-valued functions in which the dependent variable is derived by simple algebraical operations from the independent variable.

All the functions mentioned above are single-valued, that is, there is a strict one-one correspondence between the independent and dependent variables. An exception may suggest itself in the case of the square root ; in its most general form, corresponding to any admissible value of $x$ we have two values of the square root,

namely, $\pm \sqrt{x}$; but it is usual in the table to give only $+ \sqrt{x}$, to which the name of principal value is attached.

The number of the entries in a table is necessarily restricted; but there are two other kinds of restriction in constructing a table of a function which may be noted, the first of these results from the nature of the function, the second is imposed by practical utility. Thus, the square roots of negative numbers cannot be given; negative values of $x$ in the table for $\sqrt{x}$ do not therefore occur; but the squares of negative numbers are not given in the table for $x^2$, because they can be deduced immediately from the results given for positive values of $x$. Still another restriction of a different kind occurs in the case of such a function as

$$\frac{x^2 - 2}{x + 1}$$

The table of this function is, to give a few values,

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\frac{x^2 - 2}{x + 1}$ | $-2$ | $-\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{7}{4}$ | $\frac{14}{5}$ |

but no value can be given to the function when $x = -1$. In this function a single isolated value of $x$ occurs for which there is no corresponding value of the dependent variable. At a later stage this exception will be considered.

The tables of trigonometrical functions are not extended beyond the first quadrant; extension being unnecessary, as simple formulae exist which enable us to calculate the values of the ratios of angles in the second, third and fourth quadrants when we know those in the first. The formulae in the case of the sine and tangent function are

$$\sin(90^\circ + x) = \sin(90^\circ - x) \qquad \sin(180^\circ + x) = -\sin x$$
$$\sin(270^\circ + x) = -\sin(90^\circ - x)$$

$$\tan(90^\circ + x) = -\tan(90^\circ - x) \qquad \tan(180^\circ + x) = \tan x$$
$$\tan(270^\circ + x) = -\tan(90^\circ - x)$$

The use of 10 as a base of logarithms allows us to print in a small compass a large amount of information, as in this system the logarithms of positive numbers with the same digits all have the same mantissa. There is in the logarithm table also a natural restriction, as $\log x$ has no arithmetical value when $x$ is zero or negative.

A further feature of mathematical tables is that they are usually calculated to 4, 5, 7 places of decimals, or in rare cases to 12 or 15

places; the character of the work determines the kind of table which is most suitable. But it is worthy of note that in tables of surds, logarithms and trigonometrical functions the numbers corresponding to rational values of $x$ are in general irrational; no table can, therefore, give more than approximate values, unless in very exceptional cases; the only instances of rational results in the tables of trigonometrical functions are afforded by $\sin 30°$, $\cos 60°$, $\tan 45°$, and when the angle is zero or $90°$. This does not imply that there are not angles whose trigonometrical functions are rational; these must occur, but the ratio of such an angle to a right angle is certainly incommensurable, and will not be included in tables calculated on the ordinary basis of angular measurement.

The word function is often used in cases in which no mathematical process can be laid down for establishing a correspondence between two classes of numbers given by observation or experiment. But when the observations are sufficient to justify the formulation of a functional relation underlying the two classes of numbers, the observer is justified in regarding his observations as governed by a law. Such a correspondence is known to physicists to exist between the numbers denoting the pressure and the volume of a gas at constant temperature and within certain ranges of pressure. The particular function which expresses this relation in its mathematical form is known as Boyle's Law.

**3. Polynomial functions.**

It is not possible to make an exhaustive classification of functions such as we attempt in the case of numbers. But certain functions can be conveniently grouped together. The first group we consider is that of *rational algebraic functions,* in which the mathematical process by which the function is derived consists of a finite number of the four fundamental operations—addition, subtraction, multiplication and division—performed upon $x$ and upon certain constant numbers, $a$, $b$, ... .

Thus the *linear* function is defined as the result of multiplying $x$ by $a$ and adding $b$, where $a$ and $b$ are constant. The linear function implies a correspondence between two classes of numbers which are typically written

$$\begin{array}{c} x \\ ax + b \end{array}$$

As instances of linear functions we take

$$2x \qquad x - 1 \qquad -3x - 4 \qquad \tfrac{1}{2}x + 2$$

Again, if we multiply a linear function by $x$ and add a constant $c$, we obtain

$$x(ax + b) + c = ax^2 + bx + c$$

the typical form of the *quadratic function.*

Proceeding in the same way, we obtain from the quadratic function the *cubic function*

$$x(ax^2 + bx + c) + d = ax^3 + bx^2 + cx + d$$

These functions are instances of the rational integral algebraic function of $x$, $ax^n + bx^{n-1} + cx^{n-2} + \dots + hx + k$

in which $a, b, c, \dots k$ are the constant coefficients of the $n + 1$ terms, $k$ being called the constant term. This is briefly described as the *polynomial function* of the $n$th degree.

Another class of functions is obtained by taking the quotient of two polynomials; such a quotient is called a rational algebraic function of $x$.

**4. Notation of functions.**

Just as it is convenient to have a single symbol $x$ which stands for any number of a given class, so it is convenient to have a symbol which expresses any one of the functions described above, or indeed any function which comes under the general definition. The symbol used is $f(x)$; it implies some rule involving mathematical operations with numbers which allows us, when a value of $x$ is assigned, to give the corresponding value of $f(x)$.* Reverting to the definition of a single-valued function, the reader will see that the two corresponding classes of numbers which constitute the function are expressed now by the pair of symbols

$$\begin{matrix} x \\ f(x) \end{matrix}$$

a correspondence which in its expanded form may include such a table as

| | | | | |
|---|---|---|---|---|
| $-2$ | $-1$ | $0$ | $1$ | $2$ |
| $f(-2)$ | $f(-1)$ | $f(0)$ | $f(1)$ | $f(2)$ |

The second class of numbers is expressed by the symbol $y$; with this notation the correspondence which constitutes the functional relation is written

$$\begin{matrix} x \\ y \end{matrix}$$

Both notations are used continually. The second possesses an apparent defect, inasmuch as the correspondence between a particular pair of values is not indicated in the notation; it has, however, the advantage of indicating the two corresponding classes each by a single letter. There are devices which allow us to remove

* The reader may compare the symbols of $\sin(x)$, $\tan(x)$, $\log(x)$ with $f(x)$. The brackets round $x$, which are usually omitted in writing $\sin x$, $\tan x$, $\log x$, are retained in $f(x)$.

the apparent defect just indicated; one is the device of attaching the same numerical suffix to corresponding $x$'s and $y$'s. Thus the extended table of the general function can be written

| | | | | | |
|---|---|---|---|---|---|
| $x$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_n$ |
| $y$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_n$ |

where the suffix indicates the correspondence between the pairs of mated variables. These marks justify no inference as to numerical magnitude, $x_2$ may be greater or less than $x_1$; they are simply distinguishing marks similar to ma., mi., tert., by which boys in a school who have the same surname are sometimes distinguished. A similar notation of dashes is also used, but in this subject the dashes are reserved for another purpose.

The union of the two notations just explained leads to the relation

$$y = f(x)$$

It is perhaps convenient to remind the reader that of the two classes of numbers the primary class is that which consists of the values of the independent variable $x$, and the secondary class that which consists of the dependent variable $y$.

**5. Graphs of mathematical functions.**

The preceding brief sketch of a function will have suggested that, at any rate, certain functions may be represented by means of curves. The notation of $x$, $y$ for the expression of a function has reminded the reader of Cartesian coordinates. In the system devised by Descartes every point on a plane is represented by two numbers which express the position of the point relative to two (rectangular) axes in a plane, the determining numbers being denoted by $x$ (the abscissa) and $y$ (the ordinate). If now we think of a pair of corresponding values of the variables $x$, $y$ connected by a functional relation and also mark on a paper the point whose position is determined by these values considered as coordinates, we have a method of representing each corresponding pair of values of the two classes of numbers in the table of a function by means of a point. The table on p. 3 can now be extended by writing in a third row the letters corresponding to the points determined by the numbers above it. Thus the new form of the table for the square-function is written as

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
| $x^2$ | 4 | 1 | 0 | 1 | 4 | 9 |
| | $A$ | $B$ | $O$ | $C$ | $D$ | $E$ |

In this table we indicate that $A$ is the point whose coordinates are $(-2, 4)$, $B$ is $(-1, 1)$, $C$ is $(1, 1)$, $D$ is $(2, 4)$ and $E$ is $(3, 9)$. Thus, upon the squared paper we have, see Fig. 1, six points which correspond to the six values of the function given in the table. If the number of pairs of values of the function be increased by taking, for instance, a sequence of values of $x$ increasing by tenths, we obtain between the successive pairs of points nine new points. In this way the student will probably have no difficulty in con-

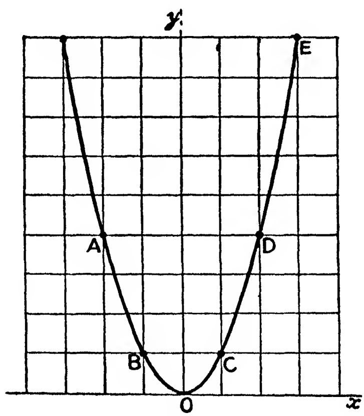



FIG. 1.

vincing himself that the graph which corresponds to a square-function is a curve. The curve is called a parabola, and in the language of analytical geometry its equation is said to be

$$y = x^2$$

The same fact is often expressed by saying that the parabola is the graph of the function $x^2$.

The student is expected to know that the graphs of linear functions are straight lines, and that those of quadratic functions are parabolas which have their axes parallel to $Oy$; the different lines which correspond to the different linear functions obtained by changing the constants $a$ and $b$ in $ax + b$ and the various parabolas which correspond to the functions $ax^2 + bx + c$ will be discussed below.

The graph of a function presents to us a rough epitome of the chief properties of the function : thus, a sine-curve suggests periodicity, while the parabola is the type of a curve with either a single maximum or a single minimum. The table of the sine-function extends only from 0° to 90° and cannot suggest any property except that of a function continuously increasing over its range of tabulation; the table of squares again does not present the principal

property of the function $x^2$, because it is only constructed for positive values of $x$. On the other hand, the rough graph of the sine-function cannot take the place of the table of sines even when the table is constructed only for each minute of angle. The truth is that the graph and the table play different parts in moulding our conception of a function. The discussions in this book require us to consider the table in its widest extent as the fundamental basis of the function, but invaluable help is often derived intuitively from the graph. By itself the table of a function suggests that the function consists of a number of pairs of values, a very large number, but still a limited number; the graph corrects this view and suggests that the number of corresponding pairs of values is unlimited. The graph does more, for it implies the converse, namely that to each point on the curve there corresponds a pair of values of $x$ and $y$.

**6. Other functions.**

Most of the graphs with which the student has to do consist of smooth or regular curves which can be traced through a certain number of guiding points by means of a flexible ruler. But there are functions which are incapable of representation in this way.

Suppose that we consider the price at different times of a loaf of bread of given weight. Here time is the abscissa or independent variable ($x$), and corresponding to each value of $x$ there is a value of the dependent variable, an ordinate ($y$), which is the price of the loaf of given weight. But as the price can only be changed by a multiple of the smallest coin of the realm (or, at any rate, by a sub-multiple of it), the graph must consist of finite straight lines parallel to the axis of $x$, the length of each line representing the interval during which the price remains unchanged. To draw a smooth curve through two points whose ordinates correspond to 3d. and $3\frac{1}{2}$d. would imply that the price of a loaf could be and was $\pi$d., a sum which cannot be liquidated satisfactorily in any existing monetary scheme.

An example of a non-mathematical function which can be represented by a smooth graph is afforded by the relation between the weight ($y$) of a man and his age ($x$). For, in this case, although our scales cannot record such a weight as $4\sqrt{5}$ stone, yet we know that if a man ever scales 9 stone there must have been a time at which he weighed $4\sqrt{5}$ stone. A smooth curve drawn through a certain number of points whose coordinates are ($x$, $y$) would give a satisfactory representation of the man's weight.

**7. A power-function.**

The graph of $2^x$, see Fig. 2, is of importance in itself and because it enforces the distinctions between different classes of numbers which were made at the beginning of the chapter. First, we consider

positive integral values of $x$ and obtain a sequence of isolated points on the graph whose coordinates are

$$(1, 2) \quad (2, 4) \quad (3, 8) \quad (4, 16) \quad (5, 32) \ldots$$

Secondly, we take a sequence of negative integral values of $x$ and obtain

$$(-1, \tfrac{1}{2}) \quad (-2, \tfrac{1}{4}) \quad (-3, \tfrac{1}{8}) \quad (-4, \tfrac{1}{16}) \quad (-5, \tfrac{1}{32}) \ldots$$

A gap between the two sequences of points is filled by defining $2^0$ as equal to 1. The next step is to give $x$ a fractional value $m/n$, where $m$ and $n$ are positive integers, and to define $2^{m/n}$ as $(\sqrt[n]{2})^m$. Thus, we may interpolate nine values between 0 and 1 from the following table

| $x$ | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 |
|---|---|---|---|---|---|---|---|---|---|
| $2^x$ | 1·07 | 1·15 | 1·23 | 1·32 | 1·41 | 1·52 | 1·62 | 1·74 | 1·87 |

It would be possible to give a great many other pairs of values, but the results given will probably indicate sufficiently well the course of a smooth curve which passes through the given points.

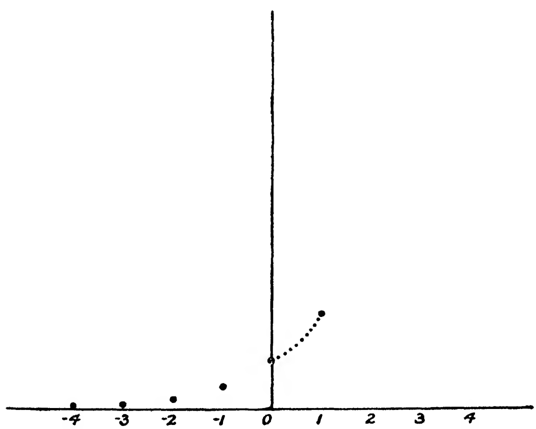



Fig. 2.

But it would be an error to suppose that the complete curve was given by taking only fractional values of $x$; this consideration takes no account of such a point as

$$(\sqrt{2},\ 2^{\sqrt{2}})$$

which certainly forms part of the complete graph. It may perhaps be added that the meaning assigned to $2^x$, when $x$ is irrational, is based upon the fact that we seek to render the complete graph of

$2^x$ a smooth curve ; the student may note that the various extensions of the meaning of $2^x$ as $x$ became negative, zero and fractional, result in the production of a graph which has the characteristic of being a smooth curve.

**8. Graph of the linear function.**

Writing $f(x) = ax + b$, it is obvious that

$$f(0) = b \qquad f(1) = a + b \qquad f(2) = 2a + b \ldots$$

This sequence of values indicates that the points which correspond to $x = 0, 1, 2, \ldots$ lie upon a straight line whose gradient is $a$. This graph is studied so fully in books on Algebra that it is sufficient to remark that, whatever values $x$ and $h$ may have,

$$\frac{f(x + h) - f(x)}{h} = \frac{a(x + h) + b - (ax + b)}{h} = a$$

In the diagram, Fig. 3,

$MP = f(x) \quad NQ = f(x + h) \quad RQ = f(x + h) - f(x) \quad RQ/PR = a$

It is only in the linear function that this simple result holds.

If we fix $P$ and take a point $Q'$ on $PQ$, or $PQ$ produced, we have, by similar triangles, $R'$ being the intersection of $PR$ and the ordinate of $Q'$,

$$\frac{R'Q'}{PR'} = \frac{RQ}{PR} = a$$

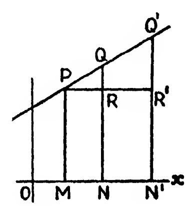



FIG. 3.

It follows that $Q'$ is on the graph, and therefore that every point of $PQ$ and of $PQ$ produced in either direction is situated upon the graph of $ax + b$.

**9. Graph of the quadratic function.**

We write $f(x) = ax^2 + 2bx + c$, the coefficient of $x$ being taken for convenience as $2b$ ; this does not imply that the coefficient is an even number, as $b$ is not necessarily an integer. The function will be discussed in a later chapter with other methods of analysis, but it is useful to resolve it now into the sum or difference of two squares. Thus,

$$\begin{aligned} f(x) &= ax^2 + 2bx + c \\ &= a\left(x^2 + \frac{2b}{a}x + \frac{c}{a}\right) = a\left(x + \frac{b}{a}\right)^2 - \frac{b^2 - ac}{a} \\ &= a\left[\left(x + \frac{b}{a}\right)^2 - \left(\frac{\sqrt{(b^2 - ac)}}{a}\right)^2\right] \\ \text{or} \qquad &= a\left[\left(x + \frac{b}{a}\right)^2 + \left(\frac{\sqrt{(ac - b^2)}}{a}\right)^2\right] \end{aligned}$$

the alternative forms being used, according as $b^2 >$ or $< ac$.

The graph of $f(x)$ is a parabola whose axis of symmetry is parallel to $Oy$. The following features of the curve are deduced from the algebraic form in which it is written.

i. The parabola is concave to an observer at a great distance above $Ox$, if $a$ is positive; convex, if $a$ is negative.

ii. The equation of the axis of symmetry is $x = - b/a$.

iii. The crest or hollow of the curve is at a distance $(ac - b^2)/a$ above the axis of $x$.

iv. If $b^2 < ac$, the parabola does not cut the axis of $x$; if $b^2 = ac$, it touches $Ox$; if $b^2 > ac$, it crosses the axis of $x$ at two points which correspond to the values of $x$ which make $f(x) = 0$, that is, at points whose abscissae are

$$-\frac{b}{a} \pm \frac{\sqrt{(b^2 - ac)}}{a}$$

It may help in drawing the parabola to notice that all quadratic functions in which $a$ has the same numerical value, regardless of sign, can be drawn with the same parabolic ruler or template. Thus, the graphs of

$$x^2 \qquad -(x-1)^2 \qquad x^2+1 \qquad 4-x^2$$

are copies of each other; they differ in their positions relative to the axes.

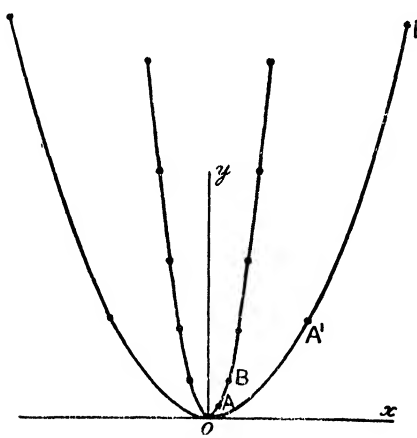



FIG. 4.

It is worth noticing, too, that all parabolas are similar, just as all circles are similar. In the diagram, Fig. 4, the parabolas

$$y = x^2 \qquad y = 10x^2$$

are drawn. The equation of the second curve may be written

$$(10y) = (10x)^2$$

a form which shows that, if the curve $y = 10x^2$ is magnified, so that the scales of $x$ and of $y$ are both increased in the ratio of 1 : 10, the magnified curve coincides with the parabola whose equation is $y = x^2$, as drawn on the original scale. In the diagram $OAB$ is the curve $y = 10x^2$, $OA'B'$ is the same curve drawn upon a magnified scale.

**10. The arithmetical continuum.**

The totality of all real numbers, however they may be classified, constitutes the arithmetical continuum. To form an adequate notion of this most important conception is difficult; it may help the student in his attempt to realise it to suggest that he should, in the first instance, concentrate attention upon that part of the continuum which lies between 0 and 1 and reflect upon the non-terminating decimal

$$0\,.\,abcdef\ldots$$

where each of the letters $a, b, c, \ldots$ stands for one of the integers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The totality of the numbers thus represented constitutes the portion of the continuum between 0 and 1. If in the general number written down above, all the digits after a certain point are zeroes, the number is a rational fraction whose denominator is of the form $2^p5^q$, where $p$ and $q$ are integers; other rational fractions are expressed by means of decimals which, after some point, recur. But, besides the class of rational fractions which is said to be *countable* because it can be arranged in a sequence, there is the larger class of incommensurable or irrational numbers which is *uncountable*, because it cannot be arranged in a sequence.

We may also briefly consider the geometrical representation of a number. On the axis of $x$ we take the points $O$ and $A$, the first of which represents zero and the second represents unity. There are elementary geometrical methods by which a point $P$ lying between $O$ and $A$, may be determined so that the ratio of $OP$ to $OA$ will represent any proper rational fraction. Every irrational number can also be represented by a point; this statement, which cannot be proved, is known as the Cantor-Dedekind postulate; its formulation constituted a valuable step in mathematical progress. To illustrate the general method of representing an incommensurable number, $\frac{1}{2}\sqrt{2}$ is taken, for which a simple direct construction is also available. The first step in the general procedure is to divide $OA$ into tenths and to take $OP_1/OA = 0{\cdot}7$, and then to divide each tenth into one hundred parts and take $P_1P_2$ as seven of these parts, so that $OP_2/OA = 0{\cdot}707$, and further to take $P_2P_3/OA = 0{\cdot}0001$, which gives $OP_3/OA = 0{\cdot}7071$. The process is as unending as the evaluation of $\sqrt{2}$ as a decimal

fraction. The Cantor-Dedekind postulate asserts that there is a point $P$ on the axis such that

$$OP/OA = \tfrac{1}{2}\sqrt{2}$$

The student to whom the process of working with sequences is new will find additional information in Chapter II. The construction of $P$ can be effected directly by drawing a square whose side is $\frac{1}{2}OA$ and marking off upon $Ox$ a line $OP$ equal to the diagonal of this square. The reader will perhaps realise better the necessity of the Cantor-Dedekind postulate by attempting the representation of such a number as $\frac{1}{4}\pi$, for which no exact geometrical construction involving a finite number of operations is available.

This process of construction for the continuum may be carried out between every consecutive pair of integers, positive or negative; in this way an idea of each section of the continuum may be formed. But there is one other question which must be considered before leaving this slight sketch, namely, What are the extremities of the continuum? An answer of a definite nature is impossible, for the continuum must contain every number which we may require to use. The extent of the continuum is therefore indefinite. An appropriate symbol to express this fact is given in the next article.

**11. Range of a variable.**

If all the numbers of the arithmetical continuum from $x = a$ to $x = b$ are taken, they represent a range of values of the variable. Now this range is called a *closed range*, if the end-values $x = a$, $x = b$, are included; it is then written $(a, b)$. But, if the end-value $x = a$ is excluded while $x = b$ is included, it is called a range open at the left end, and will be denoted by $[a, b)$; if $x = a$ is included, but $x = b$ is excluded, it is a range open at the right end, and is denoted by $(a, b]$; a range open at both ends is called an *open range*, and is denoted by $[a, b]$.

With this notation the arithmetical continuum is denoted by $[-\infty, \infty]$, the positive part of the continuum by $[0, \infty]$ and the negative part by $[-\infty, 0]$, zero being a number which is neither positive nor negative. The symbol $\infty$ may be defined for our present purposes by the statement that $[0, \infty]$ includes every positive number, however large it may be.

**12. Natural range of the independent variable of a function.**

The notation explained in Art. 11 allows us to describe succinctly the values of the variable for which a function is defined. Thus, the polynomial (or rational integral algebraic) function of $x$ is defined for every value of $x$ which lies in the range $[-\infty, \infty]$, that is, it can be calculated for any given value of $x$. The reciprocal function $1/x$ is defined for all values in the continuum, except $x = 0$; its range therefore consists of two parts, namely,

$[-\infty, 0]$ and $[0, \infty]$, for from both of these ranges $x = 0$ is excluded. The logarithmic function $\log x$ is defined for a range of $x$ denoted by $[0, \infty]$, that is, for any positive value of $x$. Such a function as $\sqrt{(a^2 - x^2)}$ is defined for values of $x$ in the range $(-a, a)$, the function being zero at the end-values of the range.

### 13. Relative magnitude of two functions.

There are six ways of expressing the relative magnitude of two functions, namely,

I. $f(x) > \varphi(x)$ (greater than)
II. $f(x) < \varphi(x)$ (less than)
III. $f(x) \ngtr \varphi(x)$ (not greater than)
IV. $f(x) \nless \varphi(x)$ (not less than)
V. $f(x) \geqq \varphi(x)$ (greater than or equal to)
VI. $f(x) \leqq \varphi(x)$ (less than or equal to)

Of these III and VI are identical, and II differs from them only in values of $x$ for which the functions are equal; again, IV and V are identical and differ from I only for those values, if any, at which equality subsists.

Each of these six relations suggests an enquiry, namely, the range of values of $x$ for which the inequality holds. In answering such queries the notation of the previous article of closed and open ranges provides an exact method of expression which the student should practise. It is often advisable to draw the graphs of the two functions; it may also be of help to determine by analysis the points of intersection of their graphs. The examples given below illustrate the two methods and their combination.

### 14. Illustrative examples.

Ex. 1. *To find the range of values of $x$ for which*

$$-2x - 3 > 3x + 2$$

We see from their graphs that

$$y = -2x - 3 \qquad y = 3x + 2$$

cross at the point $(-1, -1)$; also from this point in the direction of $x$ increasing, $y = 3x + 2$ is above $y = -2x - 3$, while on the other side of the crossing point $y = -2x - 3$ is above $y = 3x + 2$. It follows that

$$-2x - 3 > 3x + 2$$

for values of $x$ less than $-1$; the interval is open at both ends, (i) because infinity is counted as an open end, and (ii) because at $x = -1$ the functions are equal. The required range is $[-\infty, -1]$.

Ex. 2. *To discuss the values of $x$ for which*

$$6x \leqq \frac{x+6}{x+1}$$

It is easy to replace the problem given by substituting for it the inequality

$$6x - 1 \geq \frac{5}{x + 1}$$

Now the graph of $6x - 1$ is a line, while that of $5/(x + 1)$ is a rectangular hyperbola, the graphs meet at $x = \frac{2}{3}$ and at $x = -1\frac{1}{2}$, for these values of $x$ the functions are equal. The diagram, Fig. 5, in which the graphs are superposed shows that the inequality holds for two ranges

$$(-1\tfrac{1}{2}, \ -1] \text{ and } (\tfrac{2}{3}, \infty].$$

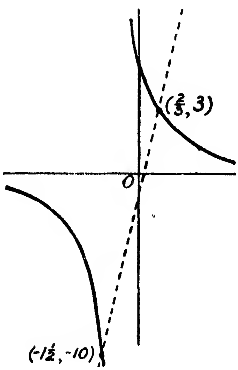



FIG. 5.

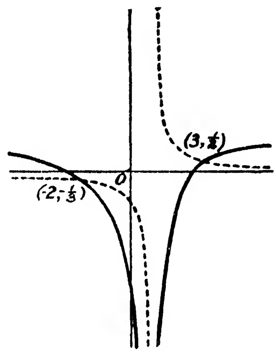



FIG. 6.

Ex. 3. *To find the ranges of $x$ for which*

$$\frac{1}{x - 1} > \frac{x^2 - 7}{(x - 1)^2}$$

The two functions are equal when $x = -2, 3$. An examination of the graphs of the functions, see Fig. 6, will convince the student that while both functions are undefined at $x = 1$, the graph of $1/(x - 1)$ is above that of $(x^2 - 7)/(x - 1)^2$ at all values of $x$ except $x = 1$ in the open range $[-2, 3]$. The inequality thus holds in $[-2, 1]$ and in $[1, 3]$. The beginner who is unpractised in drawing the graphs of such functions as are here discussed will find information on the subject in Chapter V.

As it is legitimate to multiply both sides of an inequality by a *positive* number, it might at first sight seem possible to replace the above inequality by

$$x - 1 > x^2 - 7$$

but when $x = 1$, $(x - 1)^2$ is not positive.

## EXERCISES I

1. Trace the graphs of the following linear functions of $x$

| | | |
|---|---|---|
| i. $x - 2$ | ii. $-x - 3$ | iii. $3x + 1$ |
| iv. $-2x - 4$ | v. $\frac{1}{2}x + 1$ | vi. $\frac{1}{3}(x - 2)$ |
| vii. $-\frac{1}{2}(3x - 4)$ | viii. $-\frac{1}{10}x - 1$ | ix. $10x - \frac{1}{10}$ |

2. Write down linear functions whose graphs pass through the pairs of points whose coordinates are

i. $(0, 1)\ (2, 0)$  ii. $(0, 4)\ (1, 1)$  iii. $(0, -3)\ (3, -3)$
iv. $(2, 4)\ (1, 0)$  v. $(-1, -1)\ (3, 3)$  vi. $(2, 2)\ (-2, 2)$
vii. $(4, 3)\ (3, 4)$  viii. $(-1, 5)\ (-5, 1)$  ix. $(1, -5)(2, -10)$

3. Trace the graphs of the following quadratic functions of $x$

i. $x^2 + 3$  ii. $-x^2 + 1$  iii. $\frac{1}{10}x^2$
iv. $10x^2$  v. $x(x + 3)$  vi. $(x - 4)(x + 3)$
vii. $(4 - x)(x + 3)$  viii. $x^2 + 2x + 5$  ix. $3x^2 + 6x - 1$
x. $2x^2 + 5x + 2$  xi. $x^2 - x + 1$  xii. $5x^2 - 7x + 2{\cdot}5$

4. Find quadratic functions whose graphs pass through the triads of points whose coordinates are

i. $(-2, 1)\ (0, 0)\ (2, 1)$  ii. $(0, 5)\ (1, 6)\ (3, 14)$
iii. $(-1, -1)\ (0, 1)\ (1, 1)$  iv. $(-1, 3)\ (0, 1)\ (1, 1)$
v. $(0, 0)\ (1, 3)\ (3, 15)$  vi. $(1, -2)\ (2, -8)\ (3, 12)$
vii. $(-1, -3{\cdot}1)\ (0, -2)\ (1, 0{\cdot}9)$.  viii. $(0, 2)\ (1, 0)\ (2, 6)$

5. Find the range, or ranges, of values of $x$ for which the following inequalities hold

i. $x + 1 \not> 0$  ii. $-2x - 5 > 0$
iii. $x - 3 < 2x - 4$  iv. $3x - 1 \leq 3 - x$
v. $x - 1 < 0$  vi. $3 - x < 0$
vii. $(x - 1)(3 - x) < 0$  viii. $(x + 1)(x + 2) \leq 0$
ix. $x(2x - 5) \not> 0$  x. $x/(2x - 5) < 0$
xi. $(x + 1)(3 - 2x) > 0$  xii. $6x^2 + 11x \geq 10$
xiii. $11x < 10 - 6x^2$  xiv. $6x + 11 \geq 10/x$
xv. $(x^2 - 1)/x < 0$  xvi. $3x + 2 - 1/x > 0$
xvii. $x^3 < 2x^2$  xviii. $x^4 < 2x^3$
xix. $x + \sqrt{(x + 2)} < 10$  xx. $\sqrt{(x + 1)} < \sqrt{(7 - x)}$
xxi. $\sqrt{(x + 5)} - \sqrt{x} > 1$  xxii. $3 - x > 2\sqrt{x}$
xxiii. $\dfrac{x - 1}{x + 2} < \dfrac{2x - 1}{x + 10}$  xxiv. $\dfrac{(x - 1)(x + 2)}{x(x + 3)} \geq 0$
xxv. $\dfrac{x^3 - 4}{(x - 2)^3} < \dfrac{3}{(x - 2)^2}$  xxvi. $\dfrac{x^2 + x + 11}{x^2 + 2x + 1} > 13$

6. In the sequence of rational fractions given in § 1, show that the place occupied by $a/b$ is $\frac{1}{2}(a + b - 2)(a + b - 3) + a$.

7. Show that all proper rational fractions may be arranged in a sequence. Show that in one of these sequences the place occupied by $a/b$ is $\frac{1}{2}(b - 1)(b - 2) + a$.

# CHAPTER II

## LIMIT, CONTINUITY

### 15. Irrational numbers.

We begin by examining how a physical quantity, for instance the length of a straight wire, is measured. Supposing that its length lies between 2 and 3 metres, we first mark off 2 metres from one end, and then consider the length left; if this part lies between 6 and 7 decimetres, the total length is between 2·6 and 2·7 metres. A further estimation of the part remaining after 2·6 metres have been marked off gives a length lying (say) between 2 and 3 centimetres, thus making the total length between 2·62 and 2·63 metres. With a millimetre graduation we might then get an estimate of the length as greater than 2·624 metres and less than 2·625; and with a vernier we might assign two other numbers 2·6247 and 2·6248 between which the length lies. Thus, measurement of length, and the same is true of all physical measurement, implies two sequences of numbers; these sequences are in the case of the wire

$$\begin{array}{lllll} 2, & 2{\cdot}6, & 2{\cdot}62, & 2{\cdot}624, & 2{\cdot}6247 \\ 3, & 2{\cdot}7, & 2{\cdot}63, & 2{\cdot}625, & 2{\cdot}6248 \end{array}$$

The power of extending these sequences depends upon the perfection of the observer and his instruments. But every measurement involves two such sequences, and their extension is theoretically possible.

The expression of an irrational, or incommensurable, number corresponds very closely with the above method of measuring a physical quantity, and in the case of the irrational number we are generally better able to realise the possibilities of extension of the sequences. Thus, the two sequences which define $\pi$ are

$$\begin{array}{lllllll} 3, & 3{\cdot}1, & 3{\cdot}14, & 3{\cdot}141, & 3{\cdot}1415, & 3{\cdot}14159, & 3{\cdot}141592 \\ 4, & 3{\cdot}2, & 3{\cdot}15, & 3{\cdot}142, & 3{\cdot}1416, & 3{\cdot}14160, & 3{\cdot}141593 \end{array}$$

In the case of the length of the wire and of the value of $\pi$, the sequences indicate a convenient way of separating the rational numbers into two classes, the first of which contains those which

are less than the quantity to be defined, while the second class contains those which are greater.

By a slight modification we may indicate any number N by a partition of the arithmetical continuum into two classes, one of which contains all numbers not greater than N, while the second contains all that are greater.

**16. Illustrations of a limit.**

There is one idea which must be acquired, or rather developed, by every one who studies the infinitesimal calculus; it is the conception of a limit. Before attempting to frame formal statements concerning it, the idea will be discussed in relation to two examples with which the reader is familiar.

The first illustration is drawn from recurring decimals. In arithmetic it is stated that $2{\cdot}\dot{2}$ and $2\frac{2}{9}$ are equivalent. Let us examine the meaning of this statement. Now, $2{\cdot}\dot{2}$ is an abbreviation of the non-terminating decimal

$$2{\cdot}2222\ldots$$

which, by the method of the previous article, can be defined by the two sequences

$$2,\quad 2{\cdot}2,\quad 2{\cdot}22,\quad 2{\cdot}222,\quad 2{\cdot}2222\quad \ldots\ldots \qquad \text{(A)}$$

$$3,\quad 2{\cdot}3,\quad 2{\cdot}23,\quad 2{\cdot}223,\quad 2{\cdot}2223\quad \ldots\ldots \qquad \text{(B)}$$

In working with a number which is defined by sequences, we content ourselves with the approximation suited to the purposes for which the number is required; thus, if we are satisfied with four places of decimals, we take $2{\cdot}2222$. Let us estimate the discrepancies (or errors) which occur in our work, according as we take the various terms of the sequence A instead of $2\frac{2}{9}$; these errors form a sequence which in vulgar fractions is written

$$\tfrac{2}{9},\quad \tfrac{2}{90},\quad \tfrac{2}{900},\quad \tfrac{2}{9000}\quad \ldots\ldots\ldots \qquad \text{(C)}$$

The 101st term of C is $\frac{2}{9}\,10^{-100}$, which is $< 10^{-100}$; the 101st term of A differs then from $2\frac{2}{9}$ by a quantity less than $10^{-100}$, that is, the 101st term of A agrees with $2\frac{2}{9}$ for 100 decimal places. Moreover, if we assign a magnitude $10^{-r}$, where $r$ has any integral value, however large, we may assign a term in the A-sequence, namely the $(r + 1)$th, which differs from $2\frac{2}{9}$ by less than $10^{-r}$.

Again, the errors in excess made by taking the terms of the B-sequence for $2\frac{2}{9}$ are

$$\tfrac{7}{9},\quad \tfrac{7}{90},\quad \tfrac{7}{900},\quad \tfrac{7}{9000}\quad \ldots\ldots\ldots \qquad \text{(D)}$$

and the same argument shows that the $(r + 1)$th term of the B-sequence differs from $2\frac{2}{9}$ in excess by a quantity less than $10^{-r}$.

It is thus immaterial, with the degree of accuracy which we accept, whether the $(r + 1)$th term of the A or of the B-sequence is used. From this we see that, to our degree of approximation, the numbers of either of these sequences, after a certain point, may

be replaced by $2\frac{2}{9}$. This fact is expressed otherwise by saying that $2\frac{2}{9}$ is the limit of the sequence A or of B. It is in this sense that we may say that

$$2{\cdot}2222... = 2\tfrac{2}{9}$$

The above process is illustrated in Fig. 7, in which, along an axis, we measure $OP = 2$, $PQ = 0{\cdot}2$, $QR = 0{\cdot}02$, ... and also measure $OX = 2\frac{2}{9}$. The terms of the sequence A are then represented by

$$OP, \quad OQ, \quad OR, \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A')$$

those of the sequence C by

$$PX, \quad QX, \quad RX, \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C')$$

The sequence of points $P$, $Q$, $R$, ... lies to the left of $X$ but approaches $X$, the points crowding together so closely that the geometrical treatment cannot be pursued to advantage. Now, although no point of the sequence coincides with $X$, yet if we agree to consider


FIG. 7.

two lines as equal, provided the difference in their lengths is less than any assigned small magnitude, we may say that the lengths of the lines in the A′-sequence from and after some term are equal to $OX$; for, after this term, the lengths of the corresponding lines of the C′-sequence are negligible according to the assigned standard.

The second illustration of a limit is drawn from the phenomenon of motion, as the first has been taken from counting. We shall examine the meaning of such words as "walking four miles an hour." This phrase does not imply that the pedestrian has to walk four miles, or that he has to walk for one hour in order to walk at four miles an hour. Indeed, he may have walked four miles in an hour without ever having walked at four miles an hour.

As a method of calculating velocity we take the plan which is adopted by the authorities who set traps for motorists. Two policemen, X and Y, are stationed at A and B respectively, at a measured distance apart, on a road. As the motorist passes A, X signals to Y, who starts his stop-watch; as the motor passes B, Y presses the stop-watch again and completes his observation. The time $t$ seconds registered by the watch gives the time in which the motor traverses AB, which we take as 220 yards. On the assumption of uniformity, the speed works out at

$$\frac{1}{8} \times \frac{3600}{t} = \frac{450}{t} \text{ miles per hour}$$

If there are variations in speed, the maximum speed certainly exceeds this mean speed. The single observation is sufficient for

the authorities. But if we wish to determine the velocity at A, we should have to place observers at posts nearer and nearer to A; let us suppose that the sequence of distances from A of these observers measured in yards is

$$a_1, \quad a_2, \quad a_3, \ldots$$

while the sequence of observed times in seconds is

$$t_1, \quad t_2, \quad t_3, \ldots$$

Then a sequence of observed mean speeds measured in yards per second is given by

$$\frac{a_1}{t_1}, \quad \frac{a_2}{t_2}, \quad \frac{a_3}{t_3} \ldots$$

If the observers make no errors, we have here a sequence of numbers which more and more nearly approaches the value of the velocity at A. If we are satisfied with an accuracy of one place of decimals, and find that all the terms of the sequence after a certain point agree to one place of decimals, we may take this value as the speed at A. It would be unfair to expect great accuracy from this method; the best speedometer is a very rough instrument. The existence of the above sequence and the possibility of its extension, which is implied by the dots, justifies us in speaking of the speed at A.

The speed at A is the limit of the sequence, just as $2\frac{2}{9}$ is the limit of the sequence

$$2, \quad 2{\cdot}2, \quad 2{\cdot}22, \quad 2{\cdot}222, \ldots$$

To estimate speed directly is difficult; it is generally ascertained by means of indirect methods; but to understand these methods the process of measuring speed as the limit of a sequence must be mastered.

One objection may very properly be made by the reader; he may ask why the last observation made is not sufficient. The answer to this is, that theoretically there is no last observation, for the sequence by which the speed is defined does not end. A second difficulty may arise from the fact that in the first illustration two sequences were used, and in the second only one. The reader may resolve his doubts on this point by removing the sequence B from the discussion of the recurring decimal, and thus satisfy himself that a single sequence is sufficient in defining a limit.

A distinction can be drawn between the nature of the sequences which define the recurring decimal and of that by which speed is determined. In the first case the terms of the sequence were prescribed by the nature of the decimal approximation, while in the second the sequence of speeds depended upon the arbitrary disposition of the various observers. Really the distinction does not constitute an important difference; both sequences effect a separation of the numbers of the continuum into two classes, and it is this separation, or partition, by which the number which is the limit is defined.

**17. Conditions for a limit of a sequence.**

The general typical representation of a sequence is

$$a_1, \quad a_2, \quad a_3, \ldots \qquad a_n, \ldots \qquad a_{n+m}, \ldots$$

where the suffixes define the places of the terms. We wish to find under what circumstance we are justified in saying that this sequence has a definite limit, that is, that a single number is defined by it.

Let us suppose that there are two persons engaged in settling the question; A, who wishes to establish that the sequence defines a number, and B, who has to be convinced of the validity of the process. The first step is taken by B, who agrees to lay down a scale of accuracy with which he is satisfied; he may require an accuracy corresponding to four places of decimals, he may be content with less, he may want more than four places. We shall suppose that he demands an accuracy corresponding to $r$ places of decimals. It remains now for A with this requirement before him to determine a value of $n$ such that the difference (without regard to sign) between the $n$th and any succeeding term is less than $10^{-r-1}$, that is, he has to indicate a value of $n$ such that

$$a_{n+m} \sim a_n < 10^{-r-1}$$

for all values of $m$. If he can do this, B must acknowledge that he may use $a_n$, or any succeeding term of the sequence, in work in which a degree of accuracy corresponding to $r$ places of decimals is required. The value of $a_n$ gives the value of the limit of the sequence to $r$ places of decimals. The difficulty of A's task lies in the fact that B may vary his demand by changing the value of $r$, and A has therefore to show that a value of $n$ can be assigned which satisfies the fundamental inequality, whatever the value of $r$ may be. If he does this, the sequence has a limit whose value to $r$ places of decimals is given by $a_n$.

In the sequence

$$1, \quad 1-\tfrac{1}{3}, \quad 1-\tfrac{1}{3}+\tfrac{1}{5}, \quad 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}, \ldots$$

which defines $\frac{1}{4}\pi$, a certain B carried his doubts so far as to assign to $r$ a value over 600; A, who happened to be the same person as B, successfully overcame the difficulties of the problem and evaluated $\pi$ to 616 places of decimals.

A sequence may have a rational or an irrational number as its limit. In the case of a rational limit, the fraction or integer defined by the sequence is a simpler form of expression than the sequence, but if the limit is irrational, a sequence may be the only way of expressing the number which is the limit. In either case the sequence may be regarded as determining the number which is its limit.

## 18. Example of a sequence with a rational limit.

Taking the sequence $1, \ \frac{3}{2}, \ \frac{7}{4}, \ \frac{15}{8}, \ \frac{31}{16}, \dots$

where $a_n = 2 - 2^{-n+1}$, we can show that the sequence has a limit and that this limit is 2.

First, let us find the value of $n$ which must be taken, if $r = 4$. We have

$$a_{n+m} - a_n = 2^{-n+1} - 2^{-n-m+1} = 2^{-n+1}(1 - 2^{-m})$$

Now, since $(1 - 2^{-m})$ is a proper fraction, it suffices to find a value of $n$ which makes $2^{-n+1} < 10^{-5}$. This is satisfied by $n \geqq 18$.

If we take the more general case in which $10^{-r+1}$ is assigned, we must choose $n$ so that

$$2^{-n+1} < 10^{-r}$$

that is

$$\log_{10} 2^{-n+1} < -r$$

Hence

$$n - 1 > r/\log_{10} 2$$

and

$$n > 1 + r/\log_{10} 2$$

Again, the limit of the sequence is clearly 2, because $n$ can be taken so that $a_n \sim 2$ is as small as we please.

## 19. Monotone and other sequences.

There is one class of sequences of great importance, which includes those mentioned above, namely, the sequences in which the terms either do not increase or do not diminish; such sequences are called *monotone*. As examples of such sequences, we may take

$$1, \ \tfrac{1}{2}, \ \tfrac{1}{2}, \ \tfrac{1}{4}, \ \tfrac{1}{8}, \ \tfrac{1}{8}, \dots$$
$$1, \ 2, \ 2, \ 3, \ 3, \ 3, \dots$$

The reader will be able to convince himself by studying the graphical representation of a sequence that a monotone sequence tends either to a definite limit or to infinity. The sequence of points whose coordinates are

$$(1, a_1), \quad (2, a_2), \quad (3, a_3), \dots (n, a_n), \dots$$

constitutes a graphical representation of the sequence

$$a_1, \quad a_2, \quad a_3, \dots a_n, \dots$$

Examples of monotone sequences which have an infinite limit are

$$1, \ 2, \ 3, \dots \ n, \dots$$
$$1, \ 2, \ 2^2, \dots \ 2^{n-1}, \dots$$

Non-monotone sequences which do not satisfy the test of Art. 17 are illustrated by

$$a_n = 2 + (-1)^n, \quad b_n = \sin \tfrac{1}{3} n\pi, \quad c_n = \{1 + 3(-1)^n\}^{(-1)^n}$$

The limits of the sequence $a_n$ are 3 and 1, of $b_n$ are 0, $\pm \sin\frac{1}{3}\pi$, $\pm \sin \frac{2}{3}\pi$, and of $c_n$ are 4 and $-\frac{1}{2}$. These limits are obvious on writing down a few terms of the sequences.

**20. Notation for a limit.**

If $a_n$ is the $n$th term of a sequence which has a definite limit $a$, we write

$$\lim a_n = a, \quad \text{or} \quad a_n \to a$$

$n$ being increased indefinitely.

Again, if $a_n$ is the $n$th term of a monotone sequence, we write in the case in which the sequence is increasing (or, more correctly, non-decreasing)

$$a_n \rightharpoondown a$$

while, if it is decreasing, we write

$$a_n \rightharpoonup a$$

the single barb* of the arrow indicating increase when the under barb is used, and decrease when the upper barb is drawn.

When $a_n$, increasing or decreasing, becomes greater arithmetically than any magnitude that can be assigned, we say, rather paradoxically, that the limit of $a_n$ is infinite. We distinguish (i) the case in which $a_n$ exceeds any positive magnitude that can be assigned, writing

$$a_n \rightharpoondown \infty$$

and (ii) the case in which $a_n$ is less than any negative magnitude, however large its magnitude apart from sign may be, and we write

$$a_n \rightharpoonup -\infty$$

in both cases $n$ is infinitely large, or as we may put it in this notation, $n \rightharpoondown \infty$.

**21. Discussion of f(x) in the neighbourhood of the value x = a.**

Let $a_1', a_2', a_3', \ldots$ be an ascending sequence whose limit is $a$, so that $a_n' \rightharpoondown a$; also let $a_1, a_2, a_3, \ldots$ be a descending sequence with the same limit, so that $a_n \rightharpoonup a$. Then, if $f(x)$ is defined for each number of the sequence and if

$$f(a_1'), \quad f(a_2'), \quad f(a_3'), \ldots$$

has a limit which is the same for all ascending sequences whose limit is $a$, we call this limit the left-hand limit of $f(x)$ at $x = a$, and write it

$$\mathrm{L} \lim f(x), \quad x \rightharpoondown a$$

Also if

$$f(a_1), \quad f(a_2), \quad f(a_3) \ldots$$

has a limit which is the same for all descending sequences whose limit is $a$, we call it the right-hand limit, and indicate it by

$$\mathrm{R} \lim f(x), \quad x \rightharpoonup a$$

Further, if the function is defined at $x = a$, we have a third value $f(a)$.

* The arrow notation may be read thus: $\to$ 'tends to,' $\rightharpoondown$ 'tends up to,' $\rightharpoonup$ 'tends down to.'

Various cases arise according as the magnitudes

$$\mathrm{L}\lim f(x), \quad f(a), \quad \mathrm{R}\lim f(x) \quad (x \to a)$$

are equal or not. These cases will be studied and illustrated. It is important to notice that when $f(x)$ is defined at $x = a$ and in its neighbourhood, the above three magnitudes may exist. If $f(x)$ is undefined at $x = a$, but is defined in its neighbourhood, we may then have

$$\mathrm{L}\lim f(x), \quad \mathrm{R}\lim f(x) \quad (x \to a)$$

although we cannot substitute $x = a$ in $f(x)$.

The genesis of certain types of undefined values is indicated below in Art. 23.

**22. Evaluation of f(x) when x = a, an irrational number.**

The process described in the previous article may be illustrated by the procedure which is followed sometimes for evaluating $f(x)$, when $x$ is irrational. Let us take a particular value, say, $x = \pi$ of $f(x) = (x^2 + x + 1)/(x^2 - x + 1)$. Two sequences which define $\pi$ are

$$3, \quad 3{\cdot}1, \quad 3{\cdot}14, \quad 3{\cdot}141, \quad 3{\cdot}1415, \ldots$$
$$4, \quad 3{\cdot}2, \quad 3{\cdot}15, \quad 3{\cdot}142, \quad 3{\cdot}1416, \ldots$$

In the first sequence of values $x \to \pi$, and in the second $x \to \pi$; substituting in $f(x)$, we obtain two sequences

$$f(3), \quad f(3{\cdot}1), \quad f(3{\cdot}14), \quad f(3{\cdot}141), \ldots$$
$$f(4), \quad f(3{\cdot}2), \quad f(3{\cdot}15), \quad f(3{\cdot}142), \ldots$$

Upon evaluation the following sequences for $f(\pi)$ are found

| | | | | |
|---|---|---|---|---|
| 1·857 | 1·826 | 1·8135 | 1·8132 | 1·813068 |
| 1·615 | 1·796 | 1·8107 | 1·8129 | 1·813038 |

**23. Discussion of f(x) in the neighbourhood of a value of x at which the function is undefined.**

The function $f(x)$ is undefined at $x = a$, when the mathematical processes which suffice for its determination at ordinary values fail.

Thus, for instance, if $f(x)$ is a quotient of the functions $\varphi(x)$ and $\psi(x)$ which are defined for all values in $(a, b)$, then, provided $\psi(x)$ does not vanish for any of these values, $f(x)$ may be calculated from the tables of $\varphi(x)$ and $\psi(x)$ for all values of $x$ in $(a, b)$. But if $\psi(x) = 0$ for values of $x = x_1, x_2, \ldots x_n$ in this range, then $f(x)$ cannot be calculated for any of these values of $x$, and $f(x)$ is undefined at $x = x_1, x_2, \ldots x_n$. Two cases arise which are illustrated below; in the first $\psi(x_r) = 0$, but $\varphi(x_r) \neq 0$, while in the second $\varphi(x_r) = 0$, $\psi(x_r) = 0$, $r$ being a number in the set $1, 2, \ldots n$.

The simplest case * of an undefined value is afforded by the reciprocal function $1/x$ when $x = 0$; here it is no answer to say

* Another instance is afforded by $\tan x$ $(x \to 90°)$. The erroneous statement that $\tan 90° = \infty$ is so common that the student is recommended when he has mastered

that $1/x$ is $\infty$, when $x \to 0$, the true answer being that as $x \to 0$, $1/x \to -\infty$, while as $x \to 0$, $1/x \to \infty$, and that at $x = 0$, $1/x$ is undefined. The second class of cases in which the function is undefined occurs when the function is a fraction whose numerator and denominator both vanish for $x = a$; thus $x/x^2$ is undefined at $x = 0$, and so is $x^2/x$; another instance is $\sin x/x$, when $x = 0$.

In the case when the function is undefined at $x = a$ we examine the left and right-hand limits of $f(x)$ as $x \to a$, and determine by the method of the previous articles the two magnitudes

$$\mathrm{L}\lim f(x), \quad x \to a, \quad \text{and} \quad \mathrm{R}\lim f(x), \quad x \to a$$

If both these limits are finite and are equal, we write their common value as

$$\lim f(x), \quad x \to a$$

But if one or both limits are infinite, or if they are unequal, then we can assign no value to $\lim f(x)$, $x \to a$.

**24. Continuity and discontinuity of f(x) at x = a.**

Continuity or discontinuity at $x = a$ can only occur when $x = a$ is in a range for which the function is completely defined. The conditions for continuity are that

$$\mathrm{L}\lim f(x), \quad f(a), \quad \mathrm{R}\lim f(x) \quad (x \to a)$$

are all three finite and equal.

Discontinuity occurs either when one or more of the three are infinite, or when equality does not exist between the three values.

**25. Summary with illustrations.**

A brief restatement of the method in which number and function have been regarded in this chapter will now be set forth.

Every number $(a)$, whether rational or irrational, may be approached by sequences. Thus, in order of magnitude, we have a progression of numbers

$$a_1', \quad a_2', \quad a_3', \dots a_n' \dots \qquad a, \dots a_n, \dots a_3, \quad a_2, \quad a_1 \; . \; . \quad \text{(A)}$$

If $f(x)$ is defined for every value of these two sequences, we have paired with the numbers in (A) a second set of numbers

$$f(a_1'), \quad f(a_2'), \dots f(a_n') \dots \qquad f(a), \dots f(a_n), \dots f(a_2), \quad f(a_1) \; . \quad \text{(B)}$$

Now the two sequences in (A) define the same number, whether this number is rational or irrational. But the sequences

$$f(a_1'), \quad f(a_2'), \quad f(a_3'), \dots f(a_n') \dots$$
$$f(a_1), \quad f(a_2), \quad f(a_3), \dots f(a_n) \dots$$

even when they satisfy the test of Art. 17, do not necessarily define

---

this part of the subject—perhaps the most important for his progress in the infinitesimal calculus—to examine critically the values of $\cot x$ $(x \to 0)$, $\sec x$ $(x \to 90°)$, $\operatorname{cosec} x$ $(x \to 0)$; the case of $\tan x$ is discussed on p. 30.

the same number. Supposing that the test is satisfied, the sequences define L $\lim f(x)$, $x \to a$, and R $\lim f(x)$, $x \to a$; further, we make the supposition that the same limits are found whatever sequences are taken in (A) to approach $a$.

In the case taken above $f(x)$ is defined for $x = a$, and we have three numbers to consider

$$\text{L}\lim f(x), \quad f(a), \quad \text{R}\lim f(x), \quad x \to a$$

whether $a$ is rational or irrational. But if $f(x)$ is not defined for $x = a$, but defined for values in its neighbourhood, then we have only two quantities to consider, L $\lim f(x)$, R $\lim f(x)$, $x \to a$.

The following cases arise :

I. The function may be continuous at $x = a$, in which case we have

$$\text{L}\lim f(x) = f(a) = \text{R}\lim f(x), \quad x \to a$$

all three values being finite.

II. The function may be undefined at $x = a$, and such that

$$\text{L}\lim f(x) = \text{R}\lim f(x), \quad x \to a$$

both being finite. In this case the function may be made continuous by adding to the values of $f(x)$ the value $\lim f(x)$, $x \to a$, to correspond to $x = a$.

III. The function may be discontinuous at $x = a$, when either one or more of the three quantities

$$\text{L}\lim f(x), \quad f(a), \quad \text{R}\lim f(x), \quad x \to a$$

is infinite, or two at least are unequal.

IV. The function may be undefined at $x = a$, and we may have *either*

$$\text{L}\lim f(x) \neq \text{R}\lim f(x) \quad (x \to a)$$

*or* one or both of these limits infinite.

Examples to illustrate II, III and IV are given below; the case of I is so common that it needs no special attention.

**Examples to illustrate Case II.**

Ex. 1. $$f(x) = x^2/x \quad (x \to 0)$$

It is impossible to evaluate $f(x)$, when $x = 0$. But $x^2/x = x$* for all values of $x$ except zero. Now, when $x \to 0$, $x^2/x \to 0$, and when $x \to 0$, $x^2/x \to 0$; therefore

$$\text{L}\lim x^2/x = \text{R}\lim x^2/x = 0 \quad (x \to 0)$$

although the function is undefined at $x = 0$.

* The student should notice that it is only legitimate to deduce $a = c/b$ from $ab = c$, when $b \neq 0$, so that $x = x^2/x$ follows from $x^2 = x \cdot x$, only when $x \neq 0$.

Ex. 2. $f(x) = (x + \sqrt{x} - 2)/(x - 1) \quad (x \to 1)$

The function is undefined at $x = 1$. We have

| $x$ | 1·1 | 1·01 | 1·001 | 1·0001 |
|---|---|---|---|---|
| $f(x)$ | 1·48 | 1·498 | 1·4998 | 1·49998 |

and again

| $x$ | 0·9 | 0·99 | 0·999 | 0·9999 |
|---|---|---|---|---|
| $f(x)$ | 1·51 | 1·501 | 1·5001 | 1·50001 |

From this we might infer that

$$\text{L}\lim f(x) = \text{R}\lim f(x) = 1{\cdot}5 \quad (x \to 1)$$

This result is obtained by writing

$$f(x) = \frac{(\sqrt{x} - 1)(\sqrt{x} + 2)}{(\sqrt{x} - 1)(\sqrt{x} + 1)}$$

which allows us to deduce the value of $f(x)$ for all values except $x = 1$, by substituting for $x$ in

$$\frac{\sqrt{x} + 2}{\sqrt{x} + 1}$$

Ex. 3. $f(x) = \sin x/x, \quad x \to 0$ (*$x$ being measured in degrees*).

Since $\sin(-x)/(-x) = \sin x/x$, it follows that

$$\text{L}\lim f(x) = \text{R}\lim f(x) \quad (x \to 0)$$

Also, from the table,

| $x$ | 1 | 0·8 | 0·6 | 0·4 | 0·2 | 0·1 |
|---|---|---|---|---|---|---|
| $\sin x$ | 0·0175 | 0·0140 | 0·0105 | 0·0070 | 0·0035 | 0·0017 |

we infer that $\qquad \lim \sin x/x = 0{\cdot}017 \quad (x \to 0)$

This result is confirmed by proofs given in Trigonometry, which show that

$$\lim \sin \theta/\theta = 1 \quad (\theta \to 0)$$

$\theta$ being measured in radians. The answer, when $x$ is measured in degrees, is that

$$\lim \sin x/x = \pi/180 \quad (x \to 0)$$

**Examples to illustrate Case III.**

Ex. 1. *The function $\{x\}$ which is equal to the greatest integer which is not greater than $x$.*

We have $\quad \{2\tfrac{1}{2}\} = 2, \quad \{3\} = 3, \quad \{-\tfrac{1}{2}\} = -1, \quad \{\tfrac{1}{2}\} = 0$

Now when $x$ is in the range $(0, 1]$, $\{x\} = 0$, when $x$ is in $(1, 2]$, $\{x\} = 1$, and generally when $x$ is in $(n, n + 1]$, $\{x\} = n$.

The graph of the function, see Fig. 8, consists of a succession of steps of unit length, but in each of these steps the right hand terminal point

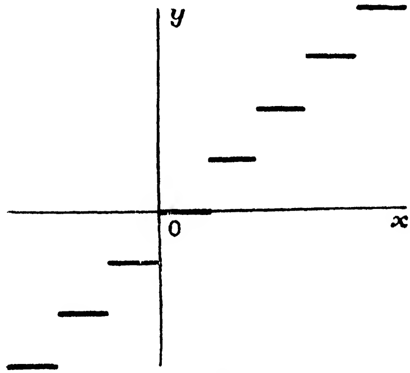



FIG. 8.

is omitted. This function provides examples of discontinuity at each integral value of $x$, for

$$\mathrm{L}\lim\{x\} = n - 1 \qquad \mathrm{R}\lim\{x\} = n \quad (x \to n) \text{ and } \{n\} = n$$

Ex. 2. *The function* M($x$), *the mantissa of* $\log_{10}x$, *and* C($x$), *the characteristic of* $\log_{10}x$.

M($x$) is discontinuous when $x = 10^n$, for

$\mathrm{L}\lim \mathrm{M}(x) = 1 \qquad \mathrm{R}\lim \mathrm{M}(x) = 0 \quad (x \to 10^n)$ and $\mathrm{M}(10^n) = 0$

Again,

$\mathrm{L}\lim \mathrm{C}(x) = n - 1 \qquad \mathrm{R}\lim \mathrm{C}(x) = n \quad (x \to 10^n)$ and $\mathrm{C}(10^n) = n$

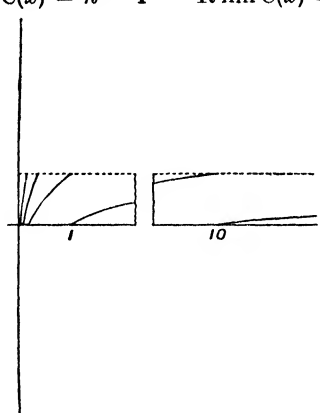



FIG. 9a.

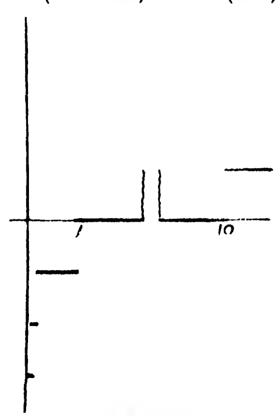



FIG. 9b.

The graph (Fig. 9$a$) of M($x$) consists of a series of lines starting from $Ox$ and just reaching $y = 1$; as the origin is approached, these lines crowd closer and closer together and become steeper and steeper.

The graph (Fig. 9b) of $C(x)$ consists of a series of steps, all on the right of $Oy$; as the axis of $y$ is approached, these steps become shorter and shorter and further and further below $Ox$. In the drawing a portion of the axis of $x$ is omitted in order to bring within the diagram an important feature of the graph.

### Examples to illustrate Case IV.

Ex. 1. $$f(x) = 1/x$$

Here we have

$$\text{L}\lim 1/x = -\infty \qquad \text{R}\lim 1/x = \infty \quad (x \to 0)$$

while $f(x)$ is undefined at $x = 0$.

Ex. 2. $$f(x) = 1/x^2 \quad (x \to 0)$$

We have $\text{L}\lim 1/x^2 = \infty \qquad \text{R}\lim 1/x^2 = \infty \quad (x \to 0)$

The function is undefined at $x = 0$.

Ex. 3. $$f(x) = \tan x \quad (x \to \tfrac{1}{2}\pi)$$

Now $\text{L}\lim \tan x = \infty \qquad \text{R}\lim \tan x = -\infty \quad (x \to \tfrac{1}{2}\pi)$

The value of $\tan x$ is undefined at $x = \frac{1}{2}\pi$.

The erroneous (but convenient) statement that $\tan \frac{1}{2}\pi = \infty$ must be regarded as an abbreviation of the statement that

$$\text{L}\lim \tan x = \infty \quad (x \to \tfrac{1}{2}\pi)$$

**26. Limit of the sum, product and quotient of two functions, when $x \to a$.**

The discussion separates itself into two parts according as the functions are defined or not for the value $x = a$.

First, we take two functions, $f(x)$ and $g(x)$, defined at $x = a$ and continuous at this value, that is,

$$\lim f(x) \to f(a) \qquad \lim g(x) \to g(a) \quad (x \to a)$$

In this case we have, as $x \to a$,

$$\begin{aligned} \lim [f(x) + g(x)] &= f(a) + g(a) = \lim f(x) + \lim g(x) \\ \lim [f(x) \,.\, g(x)] &= f(a) \,.\, g(a) = \lim f(x) \,.\, \lim g(x) \\ \lim [f(x)/g(x)] &= f(a)/g(a) = \lim f(x)/\lim g(x) \end{aligned}$$

provided that in the last case $g(a) \neq 0$.

We have also to consider the case in which the functions satisfy the following conditions, when $x \to a$,

$$\text{L}\lim f(x) = \text{R}\lim f(x) \qquad \text{L}\lim g(x) = \text{R}\lim g(x)$$

although $f(x)$, $g(x)$ (one or both) may not be defined at $x = a$. The following theorems hold

$$\begin{aligned}
\lim\,[f(x) + g(x)] &= \lim f(x) + \lim g(x) \\
\lim\,[f(x)\,.\,g(x)] &= \lim f(x)\,.\,\lim g(x) \\
\lim\,[f(x)/g(x)] &= \lim f(x)/\lim g(x)
\end{aligned}$$

provided in the last equality $\lim g(x) \neq 0$.

The proofs of these formulae present considerable difficulty; the student will do well to defer them for the present. The theorems will be frequently quoted in the following chapters. Some considerations which may assist the student in apprehending the theorems and their proof are given in Appendix I at the end of the book.

**27. Continuous functions.**

A function is continuous in a range of $x$ for which it is defined, if it is continuous for every value of $x$ in the range. Thus the polynomial function is continuous for any finite range; the reciprocal function $(1/x)$ is continuous for any finite range included in $[-\infty, 0]$ or in $[0, \infty]$; the tangent function is continuous for any range lying within any one of the ranges ... $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, $[\frac{1}{2}\pi, \frac{3}{2}\pi]$, $[\frac{3}{2}\pi, \frac{5}{2}\pi]$ ... ; while $\{x\}$ is continuous for a range included in any one of the ranges $(0, 1]$, $(1, 2]$, $(2, 3]$, ... .

It can be proved that the sum and the product of a finite number of functions which are continuous within a certain range are also continuous within this range. These theorems follow from the results quoted in Art. 26 with regard to the sum and the product of the limits of two continuous functions. It follows also that the quotient of two continuous functions is continuous, provided that the function which is the denominator does not vanish in the range considered.

Side by side with a function $f(x)$ which is undefined for some value of $x$, say $x = a$, and is yet such that

$$\text{L}\lim f(x) = \text{R}\lim f(x) = l \quad (x \to a)$$

being also continuous at all other points of its range of definition, we may construct a function which is identical with $f(x)$ at all points except $x = a$, and which at $x = a$ has the value $l$; this function may be termed an augmented $f(x)$; and we can then assert the continuity of the sum, product and quotient of two such augmented functions on the same conditions as those laid down for functions defined and continuous at every point of their range.

**28. Examples to illustrate limits.**

Ex. 1. *To find the limiting value, when $x \to 0$, of*

$$\frac{\sqrt{(1 + x + x^2)} - \sqrt{1 + x - x^2)}}{x^2}$$

Now $$\frac{\sqrt{(1+x+x^2)} - \sqrt{(1+x-x^2)}}{x^2}$$
$$= \frac{(1+x+x^2) - (1+x-x^2)}{x^2[\sqrt{(1+x+x^2)} + \sqrt{(1+x-x^2)}]}$$
$$= \frac{2}{\sqrt{(1+x+x^2)} + \sqrt{(1+x-x^2)}}$$
provided $x \neq 0$.

The denominator $\rightarrow 2$, when $x \rightarrow 0$; therefore the given fraction $\rightarrow 1$ as its limit.

As a verification we have, when $x = 0{\cdot}1$,

$$1 + x + x^2 = 1{\cdot}11 \qquad \sqrt{(1+x+x^2)} = 1{\cdot}0536$$
$$1 + x - x^2 = 1{\cdot}09 \qquad \sqrt{(1+x-x^2)} = 1{\cdot}0440$$

the value of the fraction for this value of $x$

$$= 0{\cdot}0096 \div 0{\cdot}01 = 0{\cdot}96$$

Ex. 2. *To prove that when* $n \rightarrow \infty$, $\sum_{r=0}^{n}(2r+1)/n^2 \rightarrow 1$.

Now $\sum_{r=0}^{n}(2r+1) = (n+1)^2$, by the theory of arithmetical progressions.

Therefore $$\sum_{r=0}^{n}(2r+1)/n^2 = 1 + 2/n + 1/n^2$$

and as $n \rightarrow \infty$, $1/n \rightarrow 0$ and $1 + 2/n + 1/n^2 \rightarrow 1$.

Ex. 3. *A line $BC$ of constant length slides between two fixed lines $AB$, $AC$ of unlimited length, and $B'C'$ is a second position. To find the limiting position of $P$ the intersection of $BC$ and $B'C'$, when the angle $BPB'$ decreases indefinitely.*

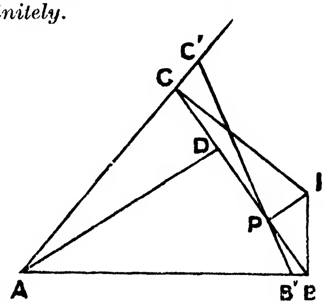



FIG. 10.

Let $AD$ be perpendicular to $BC$, and let the angle $BPB' = \delta\theta$. Then projecting $CC'B'B$ upon $CB$,

$$CB = CC' \cos(\pi - C) + C'B \cos\delta\theta + B'B \cos B$$

Also $$BB'/\sin\delta\theta = B'P/\sin B, \quad CC'/\sin\delta\theta = PC'/\sin C$$

whence $$-2CB \sin^2 \tfrac{1}{2}\delta\theta/\sin\delta\theta = PC' \cot C - B'P \cot B$$

Now when $\delta\theta \rightarrow 0$, we obtain from this

$$0 = PC \cot C - BP \cot B$$

Hence $$\frac{BP}{PC} = \frac{\tan B}{\tan C} = \frac{AD/BD}{AD/DC} = \frac{DC}{BD}$$

Also $$BP+PC = DC+BD$$

It follows that $$BP = DC \qquad PC = BD$$

*Alternative proof.*

If $BI$, $CI$ are drawn respectively perpendicular to $AB$ and $AC$ and meet in $I$, the point $I$ is called the instantaneous centre of motion; its properties are discussed in a later chapter. From the discussion given there, $IP$ is perpendicular to $BC$, and it follows that

$$BP = PI \cot PBI = PI \tan B \qquad CP = PI \tan C$$

which leads to the same result.

## EXERCISES II

**1.** Draw the graphs of the following functions

i. $\dfrac{1}{2x - 3}$ ii. $\dfrac{1}{x + 4}$ iii. $\dfrac{x}{x - 1}$

iv. $\dfrac{2x + 1}{x - 2}$ v. $\dfrac{1}{x(x - 1)}$ vi. $\dfrac{1}{x^2 - 1}$

vii. $\dfrac{1}{2x^2 + 5x + 2}$ viii. $\dfrac{1}{(2x + 5)^2}$ ix. $\dfrac{1}{x^2 + x + 1}$

comparing the graphs with the graphs of the functions which are the reciprocals of the given functions.

**2.** Determine the values of $x$ for which each of the following functions is undefined, and discuss the functions in the neighbourhood of these values

i. $1/(x + 1)$ ii. $1/\sqrt{(x + 1)}$ iii. $(x^2 + x - 1)/(x + 1)$

iv. $\dfrac{(x + 1)^2}{3x + 3}$ v. $\dfrac{x}{\sqrt{(1 - x^2)}}$ vi. $\dfrac{x^2}{(x^2 + x)^2}$

vii. $\dfrac{x - 1}{\sqrt{(x^2 - 1)}}$ viii. $\dfrac{x^3 - 3x + 2}{x^2 + 3x - 4}$ ix. $\operatorname{cosec} x$

x. $\sec^2 x$ xi. $\log \tan x$ xii. $\log \tan^2 x$

**3.** State the range, or ranges, of values of $x$ for which the following functions are defined

i. $\sqrt{(x + 1)}$ ii. $\sqrt{(x^2 - 1)}$

iii. $\sqrt{(4 - x^2)}$ iv. $\sqrt{\{(x - 1)(x - 2)\}}$

v. $[x^2(x - 1)/x]^{\frac{1}{2}}$ vi. $x^2(x - 1)/[(x - 1)^2 x]$

vii. $\sqrt{\{(x^2 - 1)/x\}}$ viii. $\sqrt{\sin x}$

ix. $\sqrt{(1 - 2 \sin x)}$ x. $\log x$

xi. $\sqrt{\log x}$ xii. $\log \tan x$

xiii. $\log \tan^2 x$ xiv. $\log x/\log(x^2 - 2x + 1)$

xv. $\log (\sin x - \cos x)$ xvi. $\log \log x$.

4. Examine the limiting values of the following functions of $x$ when $x \to 0$; the answers may be obtained by writing a sequence of small values of $x$;

i. $\dfrac{\sqrt{(1+2x)} - \sqrt{(1+x)}}{x} \to \dfrac{1}{2}$ ii. $\dfrac{1+2x^2 - \sqrt{(1+4x^2)}}{x^4} \to 2$

iii. $\dfrac{\sqrt[3]{(1+x)} - \sqrt[3]{(1-x)}}{x} \to \dfrac{2}{3}$ iv. $\dfrac{x+2-2\sqrt{(x+1)}}{x^2} \to \dfrac{1}{4}$

v. $\dfrac{(1+2x)^n - 2(1+x)^n + 1}{x^2} \to n(n-1)$, if $n > 2$

vi. $\dfrac{2^x - 1}{x} \to 0{\cdot}693$ vii. $\dfrac{3^x - 2^x}{x} \to 0{\cdot}405$

viii. $\dfrac{\sin 4x}{x} \to 4$ ix. $\dfrac{1-\cos x}{x^2} \to \dfrac{1}{2}$

x. $\dfrac{1-\cos 3x}{x^2} \to 4{\cdot}5$ xi. $\dfrac{\tan x - x}{x^3} \to \dfrac{1}{3}$

xii. $\dfrac{\tan 2x - 2x}{x^3} \to \dfrac{8}{3}$ xiii. $\operatorname{cosec} x - \cot x \to 0$

5. Prove that when $n \to \infty$ by a sequence of integers,

i. $(1+2+3+\dots+n)/n^2 \to \frac{1}{2}$
ii. $\{1+2+3+\dots+(n-1)\}/n^2 \to \frac{1}{2}$
iii. $(1^2+2^2+3^2+\dots+n^2)/n^3 \to \frac{1}{3}$
iv. $\{1^2+2^2+3^2+\dots+(n-1)^2\}/n^3 \to \frac{1}{3}$
v. $(1^3+2^3+3^3+\dots+n^3)/n^4 \to \frac{1}{4}$
vi. $(1^4+2^4+3^4+\dots+n^4)/n^5 \to \frac{1}{5}$

6. Represent graphically the sequence of points given by

$$(1, a_1) \quad (2, a_2) \quad (3, a_3) \dots (n, a_n) \dots$$

where

i. $a_n = (-1)^n/n$ ii. $a_n = n/(n+1)$ iii. $a_n = [n + (-1)^n]/n$
iv. $a_n = (1 + 1/n)^n$ v. $a_n = 1 + (-1)^n$ vi. $a_n = (-1)^n + 2^{-n}$

and determine in each case the limit, or limits, of the sequence whose $n$th term is $a_n$.

7. From a point $P$ on a quadrant of a circle, centre $O$, a perpendicular $PN$ is drawn to $OA$, one of the bounding radii, and the tangent at $P$ meets $OA$ produced in $T$. Prove that, as $P$ approaches indefinitely near to $A$,

i. $\lim PN^2/AN = 2\,OA$ ii. $\lim AN/AT = 1$
iii. $\lim PN^3/(\text{arc } AP - \text{chord } AP) = 24\,OA^2$

8. Show also with the construction of the preceding example that the limit of the ratio of the area bounded by $PN$, $NA$ and the arc $AP$ to the area of the triangle $PNT$ is $2:3$.

9. A rod slides with its ends, $A$ and $B$, one on each of two fixed axes at right angles, and $AI$, $BI$ are drawn perpendicular to the axes at $A$ and $B$. Show that the locus of a point $P$ of the rod touches the line through $P$ perpendicular to $PI$.

Prove also that the limiting position of the intersection of two consecutive positions of the rod divides $AB$ in the ratio of $OB^2 : OA^2$.

10. A line $AB$ is drawn to form with two fixed lines a triangle of constant area. Show that the limiting position of the intersection of two consecutive positions of $AB$ bisects $AB$.

11. On the rectangular axes of coordinates two points $A$, $B$ are taken, and $A$ and $B$ vary their positions slightly in such a way that the perimeter of $OAB$ is constant, the new positions being $A'$ and $B'$. Prove that

$$AA' : B'B = AB + OB : AB + OA$$

Show also that if the limiting position of the intersection of $AB$ with its consecutive position is $(x, y)$,

$$x : y = AB - OB : AB - OA$$

12. $ABC$ is an isosceles triangle whose base is $BC$; points $P$, $Q$ are taken on $CA$, $CB$ respectively so that $AP = 2\,BQ$, and $PQ$ meets $AB$ produced in $R$. Given that the limiting position of $R$ when $P$ approaches indefinitely near to $A$, is $S$, prove that

$$SB : AC = AC : 2\,BC - AC$$

# CHAPTER III

## DIFFERENTIAL COEFFICIENT

### 29. Definition.

Let $f(x)$ be a continuous function defined within an open range $[a, b]$, and let $x$ and $x + h$ be values of the variable within this range —of these two values $x$ for the present is fixed—and let

$$\frac{f(x + h) - f(x)}{h} = \mathrm{F}(h)$$

Now, for all values of $h$ such that $x + h$ is within $[a, b]$, the numerator and denominator of $\mathrm{F}(h)$ are continuous; consequently $\mathrm{F}(h)$ is continuous for every value of $h$ in $[-(x - a), b - x]$,* *except* $h = 0$; if the left and right-hand limits of $\mathrm{F}(h)$ as $h \to 0$ are equal, this common limit is called the differential coefficient of $f(x)$, and is denoted by $f'(x)$.

With the notation of the last article, we write the definition more succinctly

$$\mathrm{L}\lim\frac{f(x + h) - f(x)}{h} = \mathrm{R}\lim\frac{f(x + h) - f(x)}{h} = f'(x) \quad (h \to 0)$$

### 30. Illustrative examples.

Ex. 1. *To find the differential coefficient of* $x^2$.

Let $f(x) = x^2$.

Then $$\mathrm{F}(h) = \frac{(x + h)^2 - x^2}{h} = \frac{h(2x + h)}{h}$$

Now, $\mathrm{F}(h)$ is undefined at $h = 0$, but at other values is identical with $(2x + h)$; whence

$$f'(x) = \lim \mathrm{F}(h) \ (h \to 0) = 2x$$

Ex. 2. *To find the differential coefficient of* $x^3$.

In this case

$$\mathrm{F}(h) = \frac{(x + h)^3 - x^3}{h} = \frac{h(3x^2 + 3xh + h^2)}{h}$$

and $$f'(x) = \lim \mathrm{F}(h) \ (h \to 0) = 3x^2$$

* The end-values of the range of $h$ are obtained by giving to $x + h$ the end-values of the range $[a, b]$; thus, $x + h = a$ gives $h = -(x - a)$.

Ex. 3. *To find the differential coefficient of* $1/x$.

We have
$$F(h) = \frac{1/(x+h) - 1/x}{h} = -\frac{h}{hx(x+h)}$$

Again, $F(h)$ is undefined at $h = 0$, but at other values agrees with
$$-\frac{1}{x(x+h)}$$

it follows that
$$f'(x) = -1/x^2$$

Ex. 4. *To find the differential coefficient of* $\sqrt{x} = x^{\frac{1}{2}}$.

Here
$$F(h) = \frac{\sqrt{(x+h)} - \sqrt{x}}{h} = \frac{(x+h) - x}{h\{\sqrt{(x+h)} + \sqrt{x}\}}$$
$$= \frac{h}{h\{\sqrt{(x+h)} + \sqrt{x}\}}$$

whence
$$f'(x) = \lim F(h) \ (h \to 0) = \frac{1}{2\sqrt{x}} = \tfrac{1}{2}x^{-\frac{1}{2}}$$

Note that the range of this function is $(0, \infty]$; its range for the purpose of differentiation must be open, and is taken as $[0, \infty]$; hence the differential coefficient has no value at $x = 0$, which is confirmed by the form of $f'(x)$.

## EXERCISES III (A)

Differentiate the following functions, stating the range of values of $x$ of the differential coefficient when the range is restricted

| | | |
|---|---|---|
| 1. $2x + 3$ | 2. $x^2 - 1$ | 3. $3x^2 + x - 4$ |
| 4. $x^{-2}$ | 5. $(x+1)^{-1}$ | 6. $x^5$ |
| 7. $(2x+1)^2$ | 8. $(x^2 - 1)^{-1}$ | 9. $\sqrt{(x+1)}$ |
| 10. $\sqrt{(3 - 2x)}$ | 11. $(x^2+1)^{-2}$ | 12. $(2x+1)^{-\frac{1}{2}}$ |

### 31. Remarks upon the definition of a differential coefficient.

The left-hand limit of $F(h)$ is called the *left derivative* * of $f(x)$; also $\mathrm{R}\lim F(h)\ (h \to 0)$ is called the *right derivative*. A function may have a left and a right derivative and yet have no differential coefficient; this case occurs when
$$\mathrm{L}\lim F(h) \neq \mathrm{R}\lim F(h) \quad (h \to 0)$$

An open range was selected for the definition of $f(x)$ in order to secure that $f(x)$ might have a differential coefficient at each point of the range of its definition. If the range of definition had been closed, $f(x)$ could have had no differential coefficient at either of the closed ends; thus, if the range of $f(x)$ is $(a, b)$, $f(x)$ might have a

* The word *derivative* is not used in this book as synonymous with differential coefficient; it is only used when qualified by the words 'right' or 'left.'

right derivative at $x = a$, and a left derivative at $x = b$, but could have no differential coefficient at either end-value.

Again, the cases in which $\text{L}\lim \text{F}(h)$, $\text{R}\lim \text{F}(h)$ $(h \to 0)$ are infinite are excluded, because the equality of two infinities cannot be asserted. It follows that the differential coefficient is never infinite.

**32. Notation.**

The symbol $(')$ applied to the functional symbol, which has been adopted to indicate the differential coefficient of $f(x)$ is convenient, but other symbols are also used, and the development of the subject is bound up with the adoption of a second symbol which has proved more important than the dash.

If we write
$$y = f(x)$$
and suppose that when the independent variable is $x + \delta x$ $(= x + h)$, the dependent variable is $y + \delta y$; then we have

$$y + \delta y = f(x + \delta x) = f(x + h)$$

and
$$\text{F}(h) = \frac{f(x + h) - f(x)}{h} = \frac{\delta y}{\delta x}$$

Here $\delta x$ is a small increment of $x$ and $\delta y$ is a small increment of $y$; the limit of $\delta y/\delta x$, as $\delta x \to 0$, is the differential coefficient of $f(x)$. This limit is written
$$\frac{dy}{dx}$$

The student must note that this expression is no fraction; it is a single entity. Comparing the two notations, we have
$$y = f(x)$$
side by side with
$$\frac{dy}{dx} = f'(x)$$

**33. Geometrical illustration.**

Let $OM = x$ $\quad MP = y = f(x)$ $\quad NQ = f(x + \delta x) = y + \delta y$

Then
$$\text{F}(h) = \frac{\delta y}{\delta x} = \frac{RQ}{PR} = \tan RPQ = \tan xSP$$

Now, if we take upon the arc of the graph of the function a sequence of points $Q_1, Q_2, Q_3, \ldots$, which has $P$ as its limit, we have a sequence of chords $PQ_1, PQ_2, PQ_3, \ldots$, which, on being produced, meet the axis of $x$ in a sequence of points $S_1, S_2, S_3, \ldots$. Now the secants $Q_1PS_1, Q_2PS_2, Q_3PS_3, \ldots$ form a sequence whose limit is the tangent at $P$; hence the limit of $S_1, S_2, S_3, \ldots$ is $T$. It follows that

$$\frac{dy}{dx} = \tan xTP$$

If the tangent at $P$ is parallel to $Ox$, $S_1$, $S_2$, $S_3$ ... tend to infinity; in this case the student may replace the points of the sequence $S_1$, $S_2$, $S_3$, ... by the points in which the secants meet $Oy$.

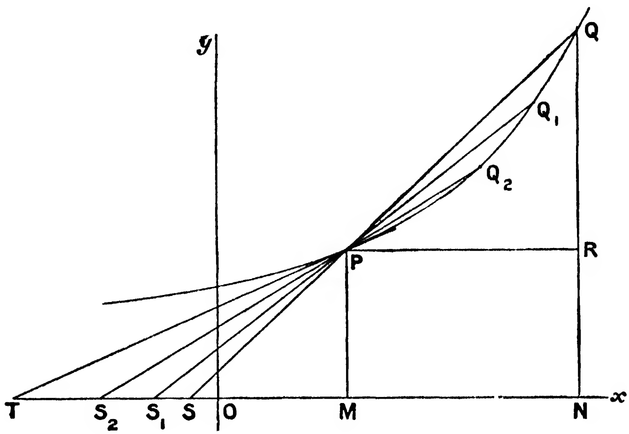


FIG. 11

**34. Second notation.**

An extension of the notation for a differential coefficient may now be made. We have

$$y = f(x) \qquad \frac{dy}{dx} = f'(x)$$

These may be replaced by

$$\frac{df(x)}{dx} = f'(x)$$

or, if we regard *differentiation* as an operation which, when applied to the function, produces its differential coefficient, we may detach the symbols and write

$$\frac{d}{dx}f(x) = f'(x)$$

This form of the notation is particularly useful when the function to be differentiated is long.

**35. Rules for differentiating the sum, product and quotient of two functions whose differential coefficients are known.**

The rules to be proved may be stated thus

I. $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

II. $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)\, g(x) + f(x)\, g'(x)$

$$= \frac{d}{dx} f(x) \cdot g(x) + f(x) \frac{d}{dx} g(x)$$

III. $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{[g(x)]^2} = \frac{\frac{d}{dx} f(x) \cdot g(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}$

In the last formula it is prescribed that $g(x)$ is not zero for the value of $x$ considered.

I. *To prove that* $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Let $s(x) = f(x) + g(x)$ ; $s(x)$ being the sum of two continuous functions, is continuous for any range for which $f(x)$, $g(x)$ are continuous.

$$s(x+h) - s(x) = f(x+h) - f(x) + g(x+h) - g(x)$$

and $$\frac{s(x+h) - s(x)}{h} = \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

Let $\mathrm{S}(h) = \frac{s(x+h) - s(x)}{h}$ $\quad \mathrm{F}(h) = \frac{f(x+h) - f(x)}{h}$

$$\mathrm{G}(h) = \frac{g(x+h) - g(x)}{h}$$

Then $$\mathrm{S}(h) = \mathrm{F}(h) + \mathrm{G}(h)$$

Here again $\mathrm{F}(h)$, $\mathrm{G}(h)$ being continuous except at $h = 0$, the same holds for $\mathrm{S}(h)$ ; also

$$\lim \mathrm{S}(h) = s'(x) \qquad \lim \mathrm{F}(h) = f'(x) \qquad \lim \mathrm{G}(h) = g'(x) \qquad (h \to 0)$$

and by the results stated in Art. 26,

$$\lim [\mathrm{F}(h) + \mathrm{G}(h)] = \lim \mathrm{F}(h) + \lim \mathrm{G}(h) \quad (h \to 0)$$

that is, $$\lim \mathrm{S}(h) = \lim \mathrm{F}(h) + \lim \mathrm{G}(h)$$

or $$s'(x) = f'(x) + g'(x)$$

This is the required theorem.

It is easy to prove the theorem for the sum of a finite number of functions, and also to show that

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

II. *To prove that*

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)\, g(x) + f(x)\, g'(x)$$

Let $p(x) = f(x)\,g(x)$, then $p(x)$ is continuous for any range for which $f(x)$, $g(x)$ are continuous; hence

$$\begin{aligned} p(x+h) - p(x) &= f(x+h)\,g(x+h) - f(x)\,g(x) \\ &= [f(x+h) - f(x)]\,g(x+h) + f(x)\,[g(x+h) - g(x)] \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{P}(h) &= \frac{p(x+h) - p(x)}{h} \\ &= \frac{f(x+h) - f(x)}{h}\,g(x+h) + f(x)\,\frac{g(x+h) - g(x)}{h} \\ &= \mathrm{F}(h)\,g(x+h) + f(x)\,\mathrm{G}(h) \end{aligned}$$

Now $\mathrm{P}(h)$ being made up of the sum of products of functions of $h$ which are continuous over ranges which do not include $h = 0$, $\mathrm{P}(h)$ is continuous except at $h = 0$. By the results of Art. 26,

$$\lim\,[\mathrm{F}(h)\,g(x+h) + f(x)\,\mathrm{G}(h)] = \lim\,[\mathrm{F}(h)\,g(x+h)] + \lim\,[f(x)\,\mathrm{G}h]$$

$$\begin{aligned} \lim \mathrm{P}(h) &= \lim \mathrm{F}(h)\,.\,\lim g(x+h) + f(x)\lim \mathrm{G}(h) \quad (h \to 0) \\ &= \lim \mathrm{F}(h)\,.\,g(x) + f(x)\lim \mathrm{G}(h) \end{aligned}$$

and, as $\lim \mathrm{P}(h) = p'(x)$ $(h \to 0)$, we have the second theorem,

$$p'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

We can deduce the differential coefficient of the product of any finite number of functions of $x$. Taking the case of three functions, and making a slight change in the notation in order to familiarise the student with the new symbols used,

$$\begin{aligned} \frac{d}{dx}(y_1y_2y_3) &= \frac{d}{dx}(y_1 \cdot y_2y_3) = \frac{dy_1}{dx} \cdot y_2y_3 + y_1\frac{d}{dx}y_2y_3 \\ &= y_2y_3\frac{dy_1}{dx} + y_1\left(y_2\frac{dy_3}{dx} + y_3\frac{dy_2}{dx}\right) \\ &= y_2y_3\frac{dy_1}{dx} + y_3y_1\frac{dy_2}{dx} + y_1y_2\frac{y_3}{dx} \end{aligned}$$

Another useful deduction is obtained by writing $g(x) = \mathrm{C}$ (a constant), in which case $g'(x) = 0$, and

$$\frac{d}{dx}[\mathrm{C}f(x)] = \mathrm{C}f'(x)$$

III. *To prove that*

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

Let $q(x) = f(x)/g(x)$, then $q(x)$ is continuous over any range for

which $f(x)$ and $g(x)$ are continuous, and in which $g(x)$ does not vanish. We have

$$q(x+h) - q(x) = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$
$$= \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x)\,g(x+h)}$$
$$= \frac{[f(x+h) - f(x)]g(x) - f(x)[g(x+h) - g(x)]}{g(x)\,g(x+h)}.$$

Again, $$\mathrm{Q}(h) = \frac{q(x+h) - q(x)}{h} = \frac{\mathrm{F}(h)\,g(x) - f(x)\,\mathrm{G}(h)}{g(x)\,g(x+h)}$$

Now $g(x) \neq 0$ and $h$ must be taken so small that $g(x+h) \neq 0$ for any small value of $h$; with these prescriptions $\mathrm{Q}(h)$ is continuous except at $h = 0$. Also when $h \to 0$,

$$\lim \frac{\mathrm{F}(h)\,g(x) - f(x)\,\mathrm{G}(h)}{g(x)\,g(x+h)} = \frac{\lim[\mathrm{F}(h)\,g(x) - f(x)\,\mathrm{G}(h)]}{g(x)\lim g(x+h)}$$
$$\lim \mathrm{Q}(h) = \frac{\lim \mathrm{F}(h)\,.\,g(x) - f(x)\lim \mathrm{G}(h)}{g(x)\,g(x)}$$

whence $$q'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

The above results are of the greatest importance and must be committed to memory. For those who may prefer words, the following statements are given:

I. *The differential coefficient of the sum of two functions is equal to the sum of their differential coefficients.*

II. *The differential coefficient of a product of two functions is the sum of the differential coefficient of the first function multiplied by the second function and the first function multiplied by the differential coefficient of the second function.*

III. *The differential coefficient of a quotient of two functions is a fraction whose numerator is the difference between the product of the denominator by the differential coefficient of the numerator and the product of the numerator by the differential coefficient of the denominator and whose denominator is the square of the denominator.*

**36. Illustrative examples.**

Ex. 1. $$\frac{d}{dx}(ax+b) = \frac{d}{dx}ax + \frac{d}{dx}b = a\frac{d}{dx}x = a$$

Ex. 2. $$\frac{d}{dx}(ax^2+bx+c) = \frac{d}{dx}ax^2 + \frac{d}{dx}bx + \frac{d}{dx}c$$
$$= a\frac{d}{dx}x^2 + b\frac{d}{dx}x$$
$$= 2ax + b$$

Ex. 3. $$\frac{d}{dx}(ax^3 + bx^2 + cx + d) = \frac{d}{dx}ax^3 + \frac{d}{dx}bx^2 + \frac{d}{dx}cx + \frac{d}{dx}d$$
$$= a\frac{d}{dx}x^3 + b\frac{d}{dx}x^2 + c\frac{d}{dx}x$$
$$= 3ax^2 + 2bx + c$$

Ex. 4.
$$\frac{d}{dx}[(ax+b)(cx+d)] = (cx+d)\frac{d}{dx}(ax+b) + (ax+b)\frac{d}{dx}(cx+d)$$
$$= (cx+d)a + (ax+b)c$$
$$= 2acx + ad + bc$$

Ex. 5. $$\frac{d}{dx}\frac{ax+b}{cx+d} = \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$
$$= \frac{a(cx+d) - c(ax+b)}{(cx+d)^2}$$
$$= \frac{ad-bc}{(cx+d)^2}$$

**37. Differential coefficient of $x^n$, when n is integral.**

First, we shall prove by the process of mathematical induction that, when $n$ is a positive integer,
$$\frac{d}{dx}x^n = nx^{n-1}$$

The theorem is true for $n = 1, 2, 3$; as we have already shown in Art. 30, from first principles, that
$$\frac{d}{dx}x = 1 . x^0 \qquad \frac{d}{dx}x^2 = 2x^1 \qquad \frac{d}{dx}x^3 = 3x^2$$

Let us assume that the theorem holds for the index $n - 1$, that is, that
$$\frac{d}{dx}x^{n-1} = (n-1)x^{n-2}$$

Then, by the result II of Art. 35,
$$\frac{d}{dx}x^n = \frac{d}{dx}x . x^{n-1} = x^{n-1}\frac{d}{dx}x + x\frac{d}{dx}x^{n-1}$$
$$= x^{n-1} + (n-1)x . x^{n-2} = nx^{n-1}$$

The theorem assumed for the $(n - 1)$th power of $x$ has been proved to be true for the $n$th, and it holds for $n = 1, 2, 3$; it therefore holds for all positive integral indices.

Secondly, let $n$ be a negative integer, say $n = -m$, where $m$ is positive; then

$$\frac{d}{dx}x^n = \frac{d}{dx}\frac{1}{x^m} = \left[x^m\frac{d}{dx}1 - 1\,.\,\frac{d}{dx}x^m\right]x^{-2m}$$

$$= \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1} = nx^{n-1}$$

The theorem holds for all integral values of $n$.

**38. The differential coefficients of sin x, cos x, tan x.**

The unit of angular measurement is the radian.

Let $f(x) = \sin x$.

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h} = \frac{2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$$

$$\mathrm{F}(h) = \cos(x+\tfrac{1}{2}h)\sin\tfrac{1}{2}h/\tfrac{1}{2}h$$

Now $\lim \mathrm{F}(h) = \lim\cos(x+\frac{1}{2}h)\lim(\sin\frac{1}{2}h/\frac{1}{2}h) \quad (h \to 0)$

$= \cos x$

Therefore $f'(x) = \cos x$ or $\dfrac{d}{dx}\sin x = \cos x$

Again, let $f(x) = \cos x$; then

$$\frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - \cos x}{h} = -\frac{2\sin(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$$

$$\mathrm{F}(h) = -\sin(x+\tfrac{1}{2}h)\sin\tfrac{1}{2}h/\tfrac{1}{2}h$$

and it follows that $\dfrac{d}{dx}\cos x = -\sin x$

Thirdly, let $y = \tan x = \sin x/\cos x$; then

$$\frac{dy}{dx} = \frac{\cos x\dfrac{d}{dx}\sin x - \sin x\dfrac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

**39. The differential coefficients of cot x, sec x, cosec x.**

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{-\sin x\sin x - \cos x\cos x}{\sin^2 x} = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x\dfrac{d}{dx}1 - \dfrac{d}{dx}\cos x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \tan x\sec x$$

$$\frac{d}{dx}\operatorname{cosec} x = \frac{d}{dx}\frac{1}{\sin x} = -\cot x\operatorname{cosec} x$$

**40. Examples on differentiation.**

Ex. 1. $$y = x \sin x$$

$$\frac{dy}{dx} = \sin x + x \cos x$$

Ex. 2. $$y = \frac{x}{\tan x}$$

$$\frac{dy}{dx} = \frac{\tan x - x \sec^2 x}{\tan^2 x} = \frac{\sin x \cos x - x}{\sin^2 x}$$

Ex. 3. $$y = \frac{x + \sin x}{x - \sin x}$$

$$\frac{dy}{dx} = \frac{(1 + \cos x)(x - \sin x) - (1 - \cos x)(x + \sin x)}{(x - \sin x)^2}$$

$$= \frac{2(x \cos x - \sin x)}{(x - \sin x)^2}$$

Ex. 4. $$y = \sin^n x$$

$$= \sin x \,.\, \sin x \ldots \sin x \ \ (n \text{ factors})$$

$$\frac{dy}{dx} = \cos x \sin^{n-1} x + \cos x \sin^{n-1} x + \ldots (n \text{ terms})$$

$$= n \cos x \sin^{n-1} x$$

Ex. 5. *Two lines $OA$, $OB$ at right angles are measured, and their lengths are given as 36, 48 yards respectively. If a possible error of 1 foot has to be allowed for in the measurement of $OA$, show that an error of 7·2 inches may occur in the length of $AB$ deduced from the measurements.*

Now, let $OA = a$, $OB = b$ and $AB = c$; then

$$c^2 = a^2 + b^2$$

If we allow an error $\delta a$ in the measurement of $a$ and a consequent error $\delta c$ in $c$, we have

$$(c + \delta c)^2 = (a + \delta a)^2 + b^2$$

On subtraction, $$2c\,\delta c + (\delta c)^2 = 2a\,\delta a + (\delta a)^2$$

or $$(2c + \delta c)\frac{\delta c}{\delta a} = 2a + \delta a$$

We want the ratio of $\delta c/\delta a$ when $\delta a$ and consequently $\delta c$ is small. This ratio is given by omitting $\delta c$ compared to $2c$ and $\delta a$ compared to $2a$, and we obtain

$$c\,\frac{\delta c}{\delta a} = a$$

whence $$\delta c = \frac{a}{c}\,\delta a$$

Substituting $a = 36$, $c = 60$, $\delta a = 12$ inches, we get the required result.

Ex. 6. *Criticise the following passage: Eridu, a port of early Babylonia, lies now* 125 *miles from the sea. If the present rate of advance (of the sea coast), about a mile in thirty years, may be taken as an average, Eridu may have been mud-bound about* 1800 B.C.

The writer assumes that by the deposit of the river Euphrates the coast has advanced at a uniform rate during 3700 years. The development of a delta is perhaps better represented by the growth of a sector of a circle; we shall calculate the rate at which the boundary of such a sector advances when the area increases at a constant rate, the angle of the sector remaining the same.

Let $A$ be the area of the sector at time $t$, $r$ its radius, $\alpha$ its angle; then

$$A = \tfrac{1}{2}\alpha r^2$$

If $\delta A$ is the increase of A, and $\delta r$ of $r$ in time $\delta t$,

$$A + \delta A = \tfrac{1}{2}\alpha(r + \delta r)^2$$

whence

$$\begin{aligned} \delta A &= \alpha r\,\delta r + \tfrac{1}{2}\alpha(\delta r)^2 \\ &= \alpha r\,\delta r, \text{ nearly} \end{aligned}$$

Now $\delta A/\delta t$ is constant, if we assume that the amount of silt is the same each year and is spread evenly over an area of equal depth. Hence $\delta r/\delta t$, the rate of advance of the sea coast, varies inversely as $r$, which means that the conclusion in the passage needs further justification.

## EXERCISES III (B)

1. Differentiate the following functions

i. $\dfrac{a - x}{a + x}$ ii. $\dfrac{x^2 - a^2}{x^2 + a^2}$ iii. $\dfrac{a + x}{a - x}$

iv. $\dfrac{x^2 + a^2}{x^2 - a^2}$ v. $\dfrac{1 + x + x^2}{1 - x + x^2}$ vi. $\dfrac{1 - x + x^2}{1 + x + x^2}$

vii. $\dfrac{1 + 3x}{1 - x}$ viii. $\dfrac{1 - 2x}{1 + x}$ ix. $\dfrac{1 + x^2}{1 - 2x + x^2}$

x. $\dfrac{1 - 2x + x^2}{1 + x^2}$ xi. $\dfrac{(x + 1)(x + 2)}{x + 3}$ xii. $\dfrac{x + 3}{(x + 1)(x + 2)}$

xiii. $\dfrac{2x - a - b}{(x - a)(x - b)}$ xiv. $\dfrac{(x + 1)(x - 2)}{(x - 3)(x + 2)}$ xv. $\dfrac{(x - 3)(x + 2)}{(x + 1)(x - 2)}$

xvi. $\dfrac{x^{2n} - a^{2n}}{x^n}$ xvii. $(1 + \sqrt{x})(1 - \sqrt{x})$ xviii. $\dfrac{1 + \sqrt{x}}{1 - \sqrt{x}}$

xix. $\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$ xx. $\sin^2 x \cos x$ xxi. $x \cos x$

xxii. $\dfrac{x}{\sin x}$ xxiii. $\dfrac{1 + \sin x}{1 - \sin x}$ xxiv. $\tan x + \sec x$

xxv. $\sec x + \operatorname{cosec} x$ xxvi. $\dfrac{\sin x - \cos x}{\sin x + \cos x}$ xxvii. $\dfrac{\sin x + \cos x}{\sin x - \cos x}$

2. Prove that the error made in estimating the length of the circumference of a circle, radius $r$, due to an error of $\delta r$ in measuring the radius is $2\pi\,\delta r$.

3. A circular track measures 398 yards close round the inside. Show that its measure 1 foot out from the inside is approximately 400 yards.

4. A square plate is contracting while cooling; and when its side is 10 inches, this side is diminishing at the rate of 0·1 inch per minute. Show that the area is decreasing at the rate of 2 sq. inches per min.

5. The radius of a circular plate is 3 inches, and is increasing at the rate of 0·01 inch per min. Show that the area is increasing at the rate of 0·19—sq. in. per min.

6. One end of a ladder, 20 feet long, rests on the ground at a distance of 12 ft. from a wall against which the other end rests. The ladder is raised, the lower end being shifted 4 in. nearer the wall. Prove that the upper end is raised about 3 in.

7. Show that the errors made in calculating (i) the surface, and (ii) the volume of a sphere, radius $r$, due to an error of $\delta r$ in the measurement of $r$, are $8\pi r\,\delta r$ and $4\pi r^2\delta r$ respectively.

If the radius is 10 in. and the error in measuring it is 0·1 in., show that the percentage errors in the surface and volume measurements are 2 p.c. and 3 p.c. respectively.

8. The volume of a cube is increasing at a constant rate. Prove that the surface-increase varies inversely as the length of the edge.

9. A rectangular block of ice rests on a non-conducting base and melts on the surface exposed to the air so that a layer of thickness 0·1 in. is converted into water each min. Show that when the height is 5 in. and the breadth and width both 3 inches, its volume is decreasing at the rate of 6·9 c. in. per min.

# CHAPTER IV

## THE SIGN OF THE DIFFERENTIAL COEFFICIENT

### 41. Derived function.

A function $f(x)$ which has a differential coefficient for every value of $x$ in an open range is said to be *differentiable* in that range. The totality of the differential coefficients of $f(x)$ constitutes another function $f'(x)$, called the *derived function* of $f(x)$.

In this chapter we propose to consider some problems relating to $f(x)$ which can be solved by a study of the properties of $f'(x)$. These problems will be discussed with, and without, the help of graphs. Graphical methods will probably appeal more directly to the beginner than those which are based on purely arithmetical considerations, but if the student wishes to obtain a firm grip of the subject, he must master the more general proofs. Possibly the beginner may be wise in deferring these to a second reading of the subject.

### 42. Meaning of the sign of $f'(x)$.

We shall show that, *if $f'(a)$ is positive, $f(x)$ is increasing as $x$ (increasing) passes through $x = a$; while if $f'(a)$ is negative, $f(x)$ is decreasing, as $x$ (increasing) passes through this value.*

We recall that in defining a differential coefficient we introduced a function

$$F(h) = \frac{f(a + h) - f(a)}{h}$$

in which the constant $a$ plays its part in the construction of $F(h)$, though it is not explicitly mentioned in the symbol $F(h)$. Further, though $F(h)$ is undefined at $h = 0$, its right and left-hand limits are equal at this value.

We now examine two ranges of the variable $h$, one on the right of $x = a$, in which case $h$ is positive, and the second on the left of $x = a$. Now, since $R \lim F(h) = f'(a)$, $h \to 0$, the terms of the sequence which gives $R \lim F(h)$ must, from and after some value of $h$, have the same sign as $f'(a)$. If then we discard terms which occur before this, we are left with a sequence of values of $h$, namely $h_1, h_2, h_3, \dots$, to which corresponds a sequence

$$F(h_1),\quad F(h_2),\quad F(h_3), \dots$$

all the terms of which have the same sign as its limit $f'(a)$. Hence, if $f'(a)$ is positive, $F(h_1)$, $F(h_2)$, $F(h_3)$, ... are all positive. It follows at once from the definition of $F(h)$ that

$$f(a + h_1) > f(a) \qquad f(a + h_2) > f(a), \dots$$

That is, when $x$ is just greater than $a$, we have

$$f(x) > f(a)$$

Again, let us approach the left-hand limit of $F(h)$ by a sequence

$$-k_1, \quad -k_2, \quad -k_3, \dots$$

such that every term of the sequence

$$F(-k_1), \quad F(-k_2), \quad F(-k_3), \dots$$

has the same sign as $f'(a)$ ; or, taking $-k$ as a typical term,

$$\frac{f(a - k) - f(a)}{-k}$$

has the same sign as $f'(a)$. Thus, if $f'(a)$ is positive, $f(a - k) - f(a)$ is negative, therefore $f(a - k) < f(a)$ ; that is, when $x$ is just less than $a$, $f(x) < f(a)$.

Joining together the two statements as to what happens just before and just after $x = a$, we see that, as $x$ passes through the value $x = a$, $f(x)$ is increasing, if $f'(a)$ is positive.

By the appropriate changes in the above statement we can show that, if $f'(a)$ is negative, $f(x)$ decreases as $x$ passes through $x = a$.

The converse of the theorem is also true, namely, that, *if $f(x)$ is increasing as $x$ passes through $x = a$, $f'(a)$ is positive* ; *while if it is decreasing, $f'(a)$ is negative*. The proof which is left to the student consists in a rearrangement of the facts stated above.

### 43. Stationary values of f(x).

If $f'(a) = 0$, we cannot say that the function is increasing or decreasing at $x = a$. When $f'(a) = 0$, we say that $f(a)$ is a stationary value, and we leave to a later section the important discussion of the various ways in which $f(x)$ may behave near such a value.

To find the stationary values of $f(x)$ we have to solve the equation $f'(x) = 0$ ; the roots of this equation determine the values of $x$ which give the stationary values of $f(x)$.

### 44. Graphical interpretation of the sign of f′(x).

The function whose graph is given in Fig. 12 is now considered. We suppose that it is described in the direction *ABC*... , in which the independent variable is increasing. From *A* to *B* the curve is on the up-gradient, and from *B* to *C* on the down-gradient, and

so on. Taking $P$ a point on the arc $AB$, and drawing the tangent at $P$, we have

$$\tan xTP = \frac{dy}{dx} \text{ (at } P) = f'(x)$$

Since $xTP$ is acute, its tangent is positive; therefore $f'(x)$ is positive. It follows that at all points on the up-gradient between $A$ and $B$, provided neither $A$ nor $B$ is included, the differential coefficient is positive.

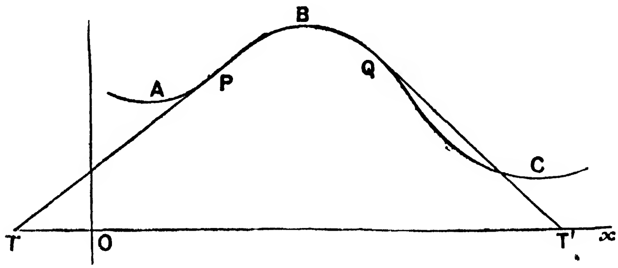



FIG. 12.

Again, at $Q$, a point on the downhill section $BC$, $xT'Q$ is obtuse and

$$\tan xT'Q = f'(x)$$

thus the differential coefficient is negative at every point on a downhill section.

The geometrical illustration presents almost intuitively the two theorems (i) that, when $f'(a)$ is positive, $f(x)$ is increasing as $x$ passes through $x = a$, and (ii) that, when $f(x)$ is increasing at $x = a$, $f'(a)$ is positive.

Now at $A$, $B$, $C$, ... , $f'(x)$ is zero, but the curve given does not afford a complete analysis of the problem of the behaviour of a curve at a stationary point.

**45. Maximum and minimum values.**

We shall discuss only maximum and minimum values which are also stationary values.

As a definition we take : *if, as $x$ passes through $x = a$, which gives a stationary value of $f(x)$, $f(x)$ changes from increasing to decreasing, then $f(a)$ is a maximum stationary value of $f(x)$ ; while, if $f(x)$ changes from decreasing to increasing, then $f(a)$ is a minimum stationary value of $f(x)$.*

In the neighbourhood of a stationary point $f'(x)$ behaves generally in one of the following ways indicated below, the exception being when $f'(x)$ remains zero on one or both sides of $x = a$ (a case which we need not particularise).

| | $x < a$ | $x > a$ | |
|---|---|---|---|
| $f'(x)$ | $+$ | $+$ | I |
| $f'(x)$ | $+$ | $-$ | II |
| $f'(x)$ | $-$ | $+$ | III |
| $f'(x)$ | $-$ | $-$ | IV |

In case I, $f(x)$ is increasing as $x$ increases to $x = a$, and goes on increasing after $x = a$, but its rate of increase is zero at $x = a$.

In case IV, a similar explanation can be given.

In case II, $f(x)$ increases as $x$ increases to $x = a$, and decreases after $x = a$; this is the case of a maximum stationary value at $x = a$.

In case III, $f(x)$ decreases as $x$ increases to $x = a$, and increases after $x = a$; this is the case of a minimum stationary value.

The geometrical interpretations of all four cases will be given below; but we can deduce that the condition for a *maximum* stationary value of $f(x)$ at $x = a$ is that, as $x$ passes through the value $x = a$, $f'(x)$ *changes from positive to negative*; while for a *minimum* stationary value, $f'(x)$ *changes from negative to positive.*

Again, if $f'(x)$ is continuous, there are no maximum and minimum values which are not stationary. For at (say) a maximum given by $x = a$, $f(x)$ changes from increasing to decreasing, and therefore $f'(x)$ changes from positive to negative; it follows, since $f'(x)$ is continuous, that $f'(a) = 0$, that is, $f(a)$ is a stationary value.

There are maximum and minimum values of a function which are not stationary values. Thus the apex $B$ of a triangle $ABC$ is the point on the line $ABC$, which is at a maximum distance from $AC$, but the perpendicular $PM$ from a point $P$ on $ABC$ is not a differentiable function of the variable $AM$ which determines its position, and therefore $PM$ has no stationary value.

### 46. Relation between the graphs of f(x) and f′(x).

This relation is so important, in view of the future use to be made of these graphs, that it is worth while side by side with the graph of $f(x)$ to draw the whole of the graph of $f'(x)$, although for the particular purpose of the problems discussed in this chapter, it is generally enough to draw only those parts of the graph of $f'(x)$ which are near to $Ox$.

In the diagram (Fig. 13) the graph of $f(x)$ is $PBCD...$, while that of $f'(x)$ is indicated by a dotted line. The points $B$, $C$ illustrate maximum and minimum values; thus $B$ indicates a maximum

stationary value, $C$ a minimum stationary, while $D$ illustrates case I of the last article and $F$ illustrates case IV; for $f'(x)$ is positive near $D'$, while $f'(x)$ is negative near $F$, vanishing at both $D'$ and $F'$.

Describing the graphs in the direction of $x$ increasing, we say that, as $P'$ is above $Ox$, $f(x)$ is increasing at $P$; as $B'$ is on the axis, $f(x)$ has a stationary value at $x = b$; again, from $B$ to $C$ $f(x)$ is decreasing, the graph of $f'(x)$ is below the axis; from $C$ to $E$, $f(x)$ does not decrease, the graph of $f'(x)$ not being below $Ox$; there is, however, a point of the graph of $f'(x)$ on the axis which corresponds to the stationary value of the function at $D$.

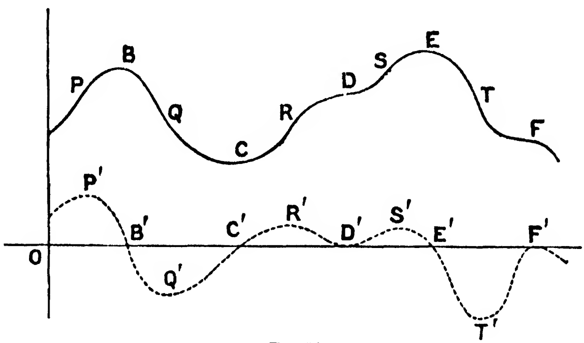



FIG. 18.

It is clear that at a maximum stationary value the graph of $f'(x)$ crosses $Ox$ from above to below; this is called (on the analogy of astronomy) a descending node; * while at a minimum stationary value the graph of $f'(x)$ crosses $Ox$ from below to above, that is, has an ascending node. At other stationary points the diagram suggests that the graph of $f'(x)$ touches $Ox$.

### 47. Examples of maxima and minima.

Ex. 1. *A quadratic function has values* 3, 5, 4, *when* $x = -1, 0, 1$ *respectively, to find its greatest value.*

Let the function be $\qquad f(x) = ax^2 + bx + c$

The conditions give

$$a - b + c = 3 \qquad c = 5 \qquad a + b + c = 4$$

These are satisfied by $a = -1\frac{1}{2}$, $b = \frac{1}{2}$, $c = 5$; hence

$$f(x) = -\tfrac{3}{2}x^2 + \tfrac{1}{2}x + 5$$
$$f'(x) = -3x + \tfrac{1}{2}$$

The stationary value is given by $x = \frac{1}{6}$; in the neighbourhood of this value, if $x < \frac{1}{6}$, $f'(x)$ is positive, while if $x > \frac{1}{6}$, $f'(x)$ is negative. It follows that $f(\frac{1}{6})$ is a maximum. The required maximum $= 5\frac{1}{24}$.

* Node implies the crossing of two lines on the diagram.

Ex. 2. *It is required to divide a line AB at P so that* $AP^2 + p \cdot BP^2$ *is a minimum.*

Let $AP = x$, $BP = a - x$.

$$f(x) = x^2 + p(a - x)^2 = x^2 + p(a^2 - 2ax + x^2)$$
$$f'(x) = 2x + p(-2a + 2x)$$
$$= 2(1 + p)x - 2ap$$

Now the graph of $f'(x)$ has an ascending node at

$$x = ap/(1 + p)$$

hence the required point of division is given by this value of $x$.

Ex. 3. *To find the stationary value of*

$$x^2 - \frac{4}{1 + x}$$

*and to show that it is a minimum.*

Let
$$f(x) = x^2 - \frac{4}{1 + x}$$
$$f'(x) = 2x + \frac{4}{(1 + x)^2} = \frac{2x(1 + x)^2 + 4}{(1 + x)^2}$$
$$= 2\frac{(x + 2)(x^2 + 1)}{(1 + x)^2}$$

The ranges in which $f(x)$ and $f'(x)$ are defined are $\lfloor -\infty, -1][-1, \infty\rfloor$. The stationary value occurs in the first of these ranges, and is given by $x = -2$; near this value, when $x < -2$, $f'(x)$ is negative, and when $x > -2$, $f'(x)$ is positive. Therefore the minimum stationary value is $f(-2) = 8$.

It is to be noted that $f(x)$ has many values less than 8 in the range $[-1, \infty]$.

Ex. 4. *The power required to propel a steamer in still water varies as the square of the speed. Prove that the most economical rate at which it can travel against a current is equal to the speed of the current.*

Let V be the rate of the current, $v$ that of the steamer relative to the land; $V + v$ is the rate relative to the water. The power varies as $(V + v)^2$, and the consumption of coal varies jointly as the power and the time; the time is equal to $l/v$, where $l$ is the length of the journey. We must therefore make $(V + v)^2/v$ a minimum.

Let
$$f(v) = (V + v)^2/v = v + 2V + V^2/v$$
$$f'(v) = 1 - V^2/v^2$$

The only values of $v$ which concern us are positive ones; $v = V$ gives a stationary value of $f(v)$, and this is a minimum, because for values of $v$ near to V which are $< V$, $f'(v)$ is negative, and for those $> V$, $f'(v)$ is positive. Hence the most economical rate of progress is when $v = V$.

Ex. 5. *To find the dimensions of an open can which is to contain* 1000 *c.cm. when the smallest amount of material is used.*

It is not inconvenient to solve first the general case. With the usual notation, we have

$$V = \pi r^2 h \qquad S = \pi r^2 + 2\pi rh = \pi r^2 + 2V/r = f(r), \text{ suppose}$$

Now
$$f'(r) = 2\pi r - 2V/r^2$$

The stationary value is given by $\pi r^3 = V$, that is, $r = (V/\pi)^{\frac{1}{3}}$. Near this value when $r$ is $< (V/\pi)^{\frac{1}{3}}$, $f'(r)$ is negative, and when $r$ is $> V(/\pi)^{\frac{1}{3}}$, $f'(r)$ is positive. Hence this value of $r$ gives the minimum of S.

The dimensions of the can of smallest surface are

$$r = (V/\pi)^{\frac{1}{3}} \qquad h = V/\pi r^2 = (V/\pi)^{\frac{1}{3}} \qquad S = 3(\pi V^2)^{\frac{1}{3}}$$

Writing $V = 1000$, $\qquad S = 439$ sq. cm.

Ex. 6. *To find the dimensions of the cylinder of greatest curved surface that can be inscribed in a sphere.*

The curved surface of the cylinder $= 2\pi rh$, and since it is inscribed in a sphere of radius $a$, $\qquad 4r^2 + h^2 = 4a^2$

Instead of finding the maximum value of S, we determine the maximum of $S^2$ or the maximum of $r^2h^2 = f(r)$, suppose. Hence

$$f(r) = r^2h^2 = 4r^2(a^2 - r^2) = 4a^2r^2 - 4r^4$$
$$f'(r) = 8r(a^2 - 2r^2)$$

Since $r$ is essentially positive, we consider only the stationary value given by $r = a/\sqrt{2}$. Now as $r$ passes through this value, $f'(r)$ changes from positive to negative. Hence, $r = a/\sqrt{2}$ gives a maximum of $f(r)$, and therefore of S.

The dimensions of the required cylinder are

$$r = a/\sqrt{2} \qquad h = a\sqrt{2} \qquad S = 2\pi a^2$$

This example can be solved also by taking an auxiliary angle $\theta$, such that $r = a \sin\theta$, $h = 2a\cos\theta$; then $S = 4\pi a^2 \sin\theta\cos\theta = 2\pi a^2 \sin 2\theta$, and the maximum of S is obviously given by $\sin 2\theta = 1$.

Ex. 7. *To find the dimensions of the right cone of greatest total surface that can be inscribed in a sphere of radius a.*

By revolution about $Ox$ the diagram determines a sphere and the inscribed cone vertex $A$. If $P$ is $(x, y)$, we have

$$(x - a)^2 + y^2 = a^2$$

The dimensions of the cone might be expressed in terms of $x$ or $y$, but it is simpler to select as variable $OP = z$; this choice is made after trial, and secures the expression of the surface as a polynomial in $z$.

Now $PN = r$, $AN = h$, $AP = s$ in the cone-notation; we have $r = y$, $h = 2a - x$, $s^2 = 2a(2a - x)$, $y^2 = x(2a - x)$, $z^2 = 2ax$, and

$$S = \pi y^2 + \pi ys$$
$$S/\pi = x(2a - x) + (2a - x)\sqrt{(2ax)}$$
$$= (2a - z^2/2a)(z^2/2a + z)$$

Differentiating,

$$\begin{aligned}\frac{d}{dz}(S/\pi) &= -z/a(z^2/2a + z) + (z/a + 1)(2a - z^2/2a)\\ &= -(2z^3 + 3az^2 - 4a^2z - 4a^3)/2a^2\\ &= -(z + 2a)(2z^2 - az - 2a^2)/2a^2\\ &= -(z + 2a)(z - z_1)(z - z_2)/a^2\end{aligned}$$

where $z_1 = \frac{1}{4}a(1 + \sqrt{17})$, $z_2 = \frac{1}{4}a(1 - \sqrt{17})$.

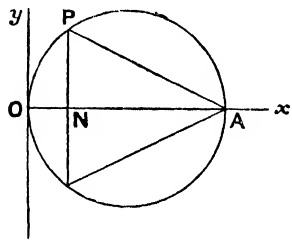



FIG 14.

Now as $z$ is essentially positive, we consider only the stationary value given by $z = z_1$; the graph of the differential coefficient shows that $z = z_1$ is a descending node: therefore $z = z_1$ gives a maximum value of S. By substitution, we find

$$\begin{aligned}r &= \tfrac{1}{16}a\sqrt{(190 + 14\sqrt{17})} = 0{\cdot}98\ldots\, a\\ h &= \tfrac{1}{16}a(23 - \sqrt{17}) = 1{\cdot}18\ldots\, a\end{aligned}$$

Ex. 8. *To find the shortest line passing through a point in the first quadrant whose coordinates are $(a, b,)$ and having its ends upon the positive parts of the axes of coordinates.*

Let $h$ and $k$ be the intercepts on the axes; we require a minimum value of $\sqrt{(h^2 + k^2)}$, where $h$ and $k$ are connected by the relation

$$a/h + b/k = 1$$

We shall seek the minimum value of

$$h^2 + k^2 = k^2 + a^2k^2/(k - b)^2 = f(k), \text{ suppose}$$

Now

$$\begin{aligned}f'(k) &= 2k - \frac{2a^2kb}{(k - b)^3}\\ &= 2k\,\frac{(k - b)^3 - a^2b}{(k - b)^3}\end{aligned}$$

The only values of $k$ which we consider lie in $[b, \infty]$; therefore we need only look at the value

$$k = b + (a^2b)^{\frac{1}{3}} = b^{\frac{1}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}})$$

It is easily seen that $f'(k)$ has an ascending node at this point. Therefore $f(k)$ is a minimum, when

$$k = b^{\frac{1}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}}) \qquad h = a^{\frac{1}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}})$$

The shortest line is $= (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

The above provides the answer to the problem of finding the longest thin beam which can be taken in a horizontal position round a rectangular bend in a passage, the widths of the two parts of the passage being $a$ and $b$. This problem is solved kinematically by noticing that in the critical position the line joining $I$, the instantaneous centre of motion of the beam, to $(a, b)$ is perpendicular to the beam. The point $I$ is discussed in a later chapter.

## EXERCISES IV

1. Find the values of $x$ for which the following functions have maximum and minimum stationary values, discriminating the two cases,

i. $3x^2 - 2x + 4$ ii. $4x^3 - 18x^2 + 15x + 7$
iii. $-x^3 + 9x^2 - 15x - 9$ iv. $x^3 - 12x^2 + 45x - 22$
v. $x^4 - 24x^2 + 64x + 5$ vi. $-x^4 + 6x^3 - 9x^2 + 4x + 12$
vii. $3x^4 + 8x^3 - 6x^2 - 24x + 30$ viii. $x^5 - 5x^4 + 5x^3 - 1$

2. State the positions of the maximum and minimum stationary values of $f(x)$ when

i. $f'(x) = x(x - 2)^2(x + 3)^3$ ii. $f'(x) = x^2(x + 2)^5$
iii. $f'(x) = x^2(x^2 + 1)^2(x - 1)^3$

3. Discuss the stationary values of

i. $(x - 2)^2(5 - 2x)$ ii. $x^2(x - 2)^2$
iii. $x^3(x + 1)$ iv. $(x - 1)^2(x + 1)^3$

4. Prove that the maximum and minimum values of a polynomial function occur alternately.

Why is not this theorem true also for the case of the quotient of two polynomials ?

5. Construct a cubic function of $x$ which has a maximum value $= 100$, given by $x = -5$, and a minimum $= -8$, given by $x = 1$.

6. Find the stationary values of the following functions, discriminating between them,

i. $x^2 + x^{-2}$ ii. $(1 - x)^3/(1 - 2x)$
iii. $\dfrac{1 - x + x^2}{1 + x + x^2}$ iv. $\dfrac{x(x^2 - 1)}{x^4 - x^2 + 1}$

7. Find the positions of the maximum and minimum stationary values of the following functions

i. $(x - 2)^2/x^3$ ii. $(x^3 - x)/(x^4 + 1)$ iii. $(x^3 + 10x)/(x^2 + 1)$

8. Determine the positions of the maximum and minimum stationary values of

i. $\sin x + \cos x$ ii. $\sin 2x + 2 \sin x$

iii. $\sin 3x - 3 \sin x$ iv. $\sin x \cos 2x$

v. $\sin^2 x \cos x$ vi. $\tan x \tan (a - x)$

9. Show that the minimum stationary value of

$$\sec x \operatorname{cosec} x + \sec x + \operatorname{cosec} x$$

is 4·83—.

10. Prove that
$$\frac{1 - 2x - x^2}{1 + x - 2x^2}$$
always decreases as $x$ increases.

11. Find the stationary value of $x^2 + y$ when $x$ and $y$ are connected by the relation
$$xy + y - 5x = 1$$
*Ans.* 13.

12. Prove that if $y$ is obtained in terms of $x$ from the equation

$$xy(y - x) = 2a^3$$

the minimum value of $y$ is $2a$.

13. A cubic function of $x$ has values 4, 2, 6, 2 when $x = 0, 1, 2, -2$ respectively. Prove that the maximum stationary value is 6.

14. A rod $AB$ is to be divided at $P$ so that $AP^2 + 3BP^2$ is a minimum. Show that $AP : PB = 3 : 1$.

15. Show that the area of the greatest rectangle that can be inscribed in a semicircle with one side upon the bounding diameter is equal to the square on the radius.

16. A point $P$ is taken upon the arc of a quadrant of a circle whose bounding radii are $OA$, $OB$, and $PM$ is drawn perpendicular to $OA$. Show that the area of the trapezium $OMPB$ is greatest when the angle $AOP$ is 30°.

17. Show that the rectangle of greatest area that can be inscribed in a triangle $ABC$ with one side lying upon $BC$ has an altitude equal to half the altitude of $ABC$.

18. A line is drawn through a fixed point, whose Cartesian coordinates are $(a, b)$, to meet $Ox$, $Oy$ in $A$ and $B$. Show that the minimum area of the triangle $OAB$ is $2ab$.

19. A rectangular strip of paper $ABCD$, whose longer sides are $AD, BC$, is folded so that the corner $A$ rests on $BC$ at $E$, and the crease meets $AB$ in $P$. Prove that the area of the triangle $EPB$ is a maximum when $BP$ is one-third of $BA$.

20. Show that the isosceles triangle of greatest area with a given perimeter is equilateral.

21. The lengths of the parallel sides of a trapezium are $a$, $a + 2x$, and the other sides are equal and of length $b$. Show that the maximum area is obtained by choosing

$$x = \tfrac{1}{4}[\sqrt{(a^2 + 8b^2)} - a]$$

22. Prove that the area of the smallest ellipse which can be drawn through the corners of a given rectangle is 1·57... times the area of the rectangle.

23. By Parcel Post regulations the sum of the length and girth of a parcel may not exceed 6 ft. Prove that (i) the largest sphere that can be sent has 1·6— c. ft., (ii) the largest cube 1·7... c. ft., (iii) the largest rectangular box of square section 2 c. ft., and (iv) the largest cylinder 2·5... c. ft.

24. From the corners of a rectangular sheet of paper, 24 in. by 9 in., four equal squares are removed, and the sides then turned up so as to form a rectangular tray. Show that when the side of each of the squares removed is 2 in., the tray has the greatest possible content.

25. A symmetrical trough is constructed having its cross-section bounded by three equal lines of given length. Prove that its capacity is greatest when the section is half a regular hexagon.

26. An open rectangular tank is to contain 288 c. feet, and the edges of its base are to be as 2 : 1. Show that the cost of lining it with lead is least when the depth is 4 feet.

27. Show that the volume of the largest cone that can be inscribed in a sphere bears to the volume of the sphere the ratio of 8 : 27.

[If $\theta$ is the semivertical angle and $\sin^2\theta = t$, the volume of the cone varies as $t(1 - t)^2$.]

28. Show that the volume of the largest cone that can be inscribed in a given cone with its vertex at the centre of the base of the given cone bears to the volume of the given cone the ratio of 4 : 27.

29. A conical tent is to have a certain content. Prove that the least amount of canvas is used when the semivertical angle is about 35° 16′.

30. A right circular cone of given volume has the smallest possible total surface. Show that the area of the curved surface is three times the area of the base.

31. A sector is cut out of a circular piece of paper, and the straight edges being joined a cone is formed. Show that the volume of this cone is a maximum when the angle of the sector removed is $2\pi(1 - \frac{1}{3}\sqrt{6})$ radians = 66° nearly.

32. Show that the cylinder inscribed in a sphere whose curved surface is greatest has the curved surface equal in area to one-half the surface of the sphere.

33. Show that in the cylinder, whose total surface is least for a given volume, the height is equal to the diameter of the base.

34. A cylinder is inscribed in a cone whose vertical angle exceeds 53° 8′. Show that the greatest value of the total surface of the cylinder is twice the area of the base of the cone, and that this is not a stationary value.

35. The base of a statue 7 feet high is 9 feet above the observer's eye. Show that the best view is obtained from a position 12 feet away.

36. Two ships, $A$ and $B$, steam the one directly towards $C$ and the other directly away, their speeds being as $2:1$, and the angle $ACB$ is $60°$. Prove that when the ships are nearest, their distances from the port are in the ratio of $4:5$.

37. A man in a boat three miles from $A$, the nearest point of a shore which is straight, wishes to reach a point on the shore which is 5 miles distant from $A$; he can walk at 5 m.p.h. and row at 4 m.p.h. Show that he should row 5 miles and walk one mile.

38. Determine the most economical speed of a steamer which costs £$a$ a day exclusive of the cost of coal, the expenditure of coal varying as the $n$th power of the speed ($n > 1$), and being £$b$ per day when the speed is $V$.

*Ans.* Required speed $= V\left[\frac{a}{(n-1)b}\right]^{\frac{1}{n}}$

39. Given that the force exerted by a circular electric current of radius $a$ on a magnet whose axis coincides with the axis of the coil varies as

$$x(a^2 + x^2)^{-\frac{5}{2}}$$

where $x$ is the distance of the magnet from the centre of the circle, show that the force is greatest when $x = \frac{1}{2}a$.

40. The strength of a rectangular beam of breadth $b$ and depth $d$ is proportional to $bd^2$ and its stiffness to $bd^3$. Show (i) that in the strongest beam that can be cut from a cylindrical trunk the perpendicular from a corner of a cross-section on the opposite diagonal divides that diagonal in the ratio of $1:2$, and (ii) that in the stiffest beam the corresponding ratio is $1:3$.

# CHAPTER V

## ALGEBRAIC FUNCTIONS

### 48. Quadratic functions.

The object of this chapter is to study $f(x)$ by the help of its derived function $f'(x)$. The first functions considered are *polynomial* functions, which are of the type

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

We shall take for detailed study the simplest polynomials, and first consider the quadratic function, which we shall write

$$f(x) = ax^2 + 2bx + c$$

its derived function being

$$f'(x) = 2ax + 2b = 2(ax + b)$$

The graph of $f'(x)$ is a line whose slope to a person describing it in the direction of $x$ increasing is up if $a$ is positive, and down if $a$ is negative. The node of $f'(x)$ is given by $x = -b/a$; it is an ascending node if $a$ is positive, and a descending node if $a$ is negative. Hence $f(x)$ has a stationary value $f(-b/a)$, which is a maximum if $a$ is negative, and a minimum if $a$ is positive. Again,

$$\begin{aligned} f(-b/a) &= ab^2/a^2 + 2b(-b/a) + c \\ &= (ac - b^2)/a \end{aligned}$$

The quantity $(ac - b^2)$ is called the *discriminant*, and is indicated by $\Delta$. The stationary value of the function is $\Delta/a$.

First, let us take *the case in which $\Delta$ is positive.* If $a$ is positive, the stationary value is positive and is a minimum. Hence the function is always positive. If $a$ is negative, the stationary value is negative and is a maximum. Hence the function is always negative. In neither case does the graph cross $Ox$. Therefore when $\Delta$ is positive, the equation $f(x) = 0$ has no real roots.

Secondly, let us take *the case in which $\Delta$ is negative.* If $a$ is positive, the stationary value is negative and is a minimum; in this case the graph cuts $Ox$ in two points, one on each side of $x = -b/a$.

If $a$ is negative, the stationary value is positive and is a maximum, and, as before, the graph cuts $Ox$ in two points.

In both cases the equation has two roots which are equidistant from $x = -b/a$.

Thirdly, *when* $\Delta = 0$. In this case the stationary value is zero, and the graph of $f(x)$ touches $Ox$ at the point where the graph of $f'(x)$ crosses it; the graph of $f(x)$ is above $Ox$ if $a$ is positive, and below if $a$ is negative. The equation $f(x) = 0$ has equal roots, namely $x = -b/a$.

The character of the function is entirely determined by the signs of $\Delta$ and $a$. This result also follows by writing the equation in the standard form of Art. 9,

$$\begin{aligned} f(x) &= a(x + b/a)^2 + (ac - b^2)/a \\ &= ax'^2 + \Delta/a \end{aligned}$$

where $x' = x + b/a$. This form of the function in which $x'$ is the new independent variable corresponds in the graph to a change of origin, the new origin from which $x'$ is measured being at a distance $b/a$ to the left of $O$. If $O'$ is the new origin, the new axis of $y$ is the axis of symmetry of the function; it is spoken of as the axis of the curve, that is, the axis of the parabola which is the graph of the quadratic function represented.

## 49. Cubic functions.

By shifting the position of the origin, the general cubic function may be reduced so that the term involving $x^2$ is wanting. This point will be illustrated sufficiently by examples. We shall also, for the sake of simplicity, consider only cubic functions in which the coefficient of $x^3$ is unity. The specialised cubic which we shall discuss is of the form

$$f(x) = x^3 - 3px + q$$

whence

$$f'(x) = 3(x^2 - p)$$

Now, if $p$ is negative the graph of $f'(x)$ does not cut $Ox$; hence, unless $p$ is zero or positive, the function has no stationary values. Taking $p$ as positive, the stationary value corresponding to $x = \sqrt{p}$ is

$$f(\sqrt{p}) = p\sqrt{p} - 3p\sqrt{p} + q = q - 2p\sqrt{p}$$

Similarly

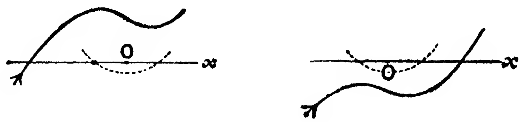
$$f(-\sqrt{p}) = q + 2p\sqrt{p}$$

Writing $\Delta = f(\sqrt{p}) \,.\, f(-\sqrt{p}) = q^2 - 4p^3$, we have a quantity which is of importance in discriminating the nature of the cubic function. For, if $\Delta$ is positive, $f(\sqrt{p})$ and $f(-\sqrt{p})$ are *either* both positive *or* both negative; while, if $\Delta$ is negative, one of the stationary values is negative and the other positive. Again, when $p$ is negative, although there are no stationary values, yet $\Delta$ exists, and in this case, as $p^3$ is negative, $\Delta$ is always positive.

We can now discuss the nature of the cubic function according as $\Delta$ is positive, negative or zero.

First, when $\Delta$ is positive.

(i) Let $p$ be positive; there are then two stationary values, and they are both positive or both negative. The graph of the function




(a) FIG. 15. (b)

has one of two forms sketched in Fig. 15. It is clear that the function vanishes *either* for a value of $x < -\sqrt{p}$, the value which gives the first stationary value arrived at as the curve is described in the direction of the arrow (Fig. 15, *a*), *or* for a value of $x$ which is greater than $\sqrt{p}$ (Fig. 15, *b*).

(ii) Let $p$ be negative; there is then no stationary value. Since $f'(x)$ is positive for all values of $x$, the function is always increasing, and its graph crosses $Ox$ only once. If $q = f(0)$ is positive, the crossing point is to the left of $O$; if $q$ is negative, it is to the right; the sign of the single real root of $f(x) = 0$ is opposite to that of $q$.

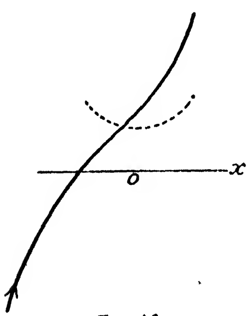



FIG. 16.

Hence, when $\Delta$ is positive the cubic equation $f(x) = 0$ has one and only one real root.

Secondly, when $\Delta$ is negative.

In this case $p$ is positive, for $q^2 - 4p^3$ is negative; and the stationary values are one positive and one negative. As in the previous case, $x = -\sqrt{p}$ is a descending node of $f'(x)$ and $x = \sqrt{p}$ is an ascending node. Therefore, $f(-\sqrt{p})$ is a maximum and $f(\sqrt{p})$ is a minimum. Again, from the values of $f(-\sqrt{p})$ and $f(\sqrt{p})$ it is obvious that

$$f(-\sqrt{p}) > f(\sqrt{p})$$

therefore $f(-\sqrt{p})$ is the positive stationary value and $f(\sqrt{p})$ the negative one. The diagram is shown in Fig. 17. In this case the graph of $f(x)$ must cross the axis three times, and the equation $f(x) = 0$ has three real roots, which are separated by the numbers $\sqrt{p}$, $-\sqrt{p}$. In other words, the roots of $f(x) = 0$ are separated by those of $f'(x) = 0$. This theorem is true for any polynomial function, and is called Rolle's Theorem (see Art. 151).

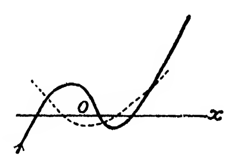



FIG. 17.

Thirdly, when $\Delta = 0$.

In this case, either the maximum or the minimum stationary value vanishes. Now, if $q$ is positive, $f(-\sqrt{p})$ cannot vanish. Therefore $f(\sqrt{p})$ must be zero; while, if $q$ is negative, $f(-\sqrt{p}) = 0$.


(a) FIG. 18. (b)

The two cases are illustrated by the left and right-hand diagrams of Fig. 18. The result is confirmed by considering the sign of $f(0) = q$.

Again, if $\Delta = 0$, $q^2 = 4p^3$, and if $q$ is positive

$$f(x) = x^3 - 3px + 2p\sqrt{p}$$
$$= (x - \sqrt{p})^2(x + 2\sqrt{p})$$

while if $q$ is negative

$$f(x) = x^3 - 3px - 2p\sqrt{p}$$
$$= (x + \sqrt{p})^2(x - 2\sqrt{p})$$

These resolutions enable us to solve the equation $f(x) = 0$ completely.

**50. The infinities of linear, quadratic and cubic functions.**

As $x$ increases beyond a certain value, the graph of a polynomial function passes beyond the limits of the paper, and it is usual to speak of the graph as proceeding to infinity. Now the ultimate parts of these curves are disposed with respect to the axes in one of four ways, which may be described by the quadrants in which the graph ultimately lies; the four alternative dispositions are

(A) in the second and first quadrants
(B) in the second and fourth quadrants
(C) in the third and first quadrants
(D) in the third and fourth quadrants

The infinite branches of the linear and cubic function are of the B or C type, and this is true of all odd polynomials; while the quadratic and other even polynomials are of the A or D type.

The above statements are inferences drawn from the graphs. The same results may be deduced by substituting large values of $x$, positive and negative, in such pairs of adjacent functions as

$$\begin{array}{ll} -2x + 3 & -2x \\ x^2 - 2x + 3 & x^2 \\ -x^3 + x - 3 & -x^3 \end{array}$$

in which the second member of each pair consists of the term containing the highest power of $x$ in the first member. If in these three pairs we write $x = 100$, we obtain

$$-197, \quad -200; \quad 9803, 10000; \quad -999903, \quad -1000000$$

values which are so nearly equal that it would be impossible without great labour to distinguish between them on a graph. The relative * differences in the various cases are all small; thus, neglecting signs, we write them as

$$\tfrac{3}{200} = 0{\cdot}015 \qquad \tfrac{197}{10000} = 0{\cdot}0197 \qquad \tfrac{97}{1000000} = 0{\cdot}000097$$

We could, however, by taking $x$ large enough, make any one of the relative differences as small as we please. To show how much smaller the relative differences become when $x$ is taken large, we may select $x = 10^6$ in the linear function, when we obtain a relative difference $= \frac{3}{2}10^{-6}$.

We will now prove that *the relative error made in taking $ax^3$ instead of $ax^3 + bx^2 + cx + d$ can be made as small as we please by taking $x$ sufficiently large.*

The relative error is

$$= (bx^2 + cx + d)/ax^3$$
$$= (b/a)x^{-1} + (c/a)x^{-2} + (d/a)x^{-3}$$

If $p$ is the greatest of the coefficients $b/a$, $c/a$, $d/a$, without regard to sign, the relative error is not greater than

$$p(x^{-1} + x^{-2} + x^{-3})$$

which again is less than $3px^{-1}$. If $\varepsilon$ is the standard value below which it is desired that our relative error should fall, the required result is obtained by taking $x = 3p/\varepsilon$.

We have proved that if, for instance, an error of 1 in 1000 is negligible, a value of $x$ can be assigned such that, for it and for greater values of $x$, no distinction need be made between the graphs of $ax^3 + bx^2 + cx + d$ and that of $ax^3$. We conclude that when $x \to \infty$, the graph of the cubic function is in the first quadrant if $a$ is positive, and in the third when $x \to -\infty$, while, in the case of $a$ negative, the graph is in the second and fourth quadrants when $x$ is very large.

### 51. Illustrative examples.

A few examples are given to show the student how important it is to associate the study of the derived function with that of the function.

* The relative, or proportional, difference between $m$ and $n$ is

$$(m - n)/m \quad \text{or} \quad (m - n)/n$$

Ex. 1.
$$f(x) = x^3 - 3x^2 - 9x + 7$$
$$f'(x) = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$$
The stationary values of $f(x)$ are given by $x = -1, 3$. Again, $x = -1$ is a descending node of $f'(x)$; hence $f(-1) = 12$ is a maximum, while $f(3) = -20$ is a minimum. The equation has three roots separated by the numbers $-1, 3$.

Ex. 2.
$$f(x) = 4x^3 - 12x^2 + 12x - 2$$
$$f'(x) = 12(x^2 - 2x + 1) = 12(x - 1)^2$$
The stationary value of $f(x)$ is given by $x = 1$, but as $f'(x)$ does not change sign, the stationary value is not a maximum or a minimum. As $f'(x)$ is never negative, $f(x)$ never decreases as $x$ increases. Again, since $f(0) = -2$, the equation has one positive root, and this root lies between 0 and 1, since $f(1) = 2$ (see Fig. 19).

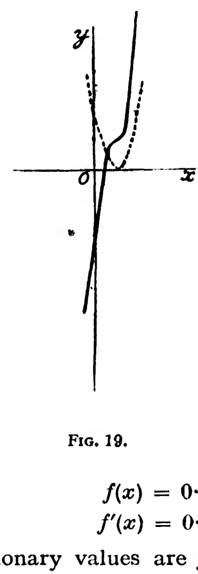


FIG. 19.

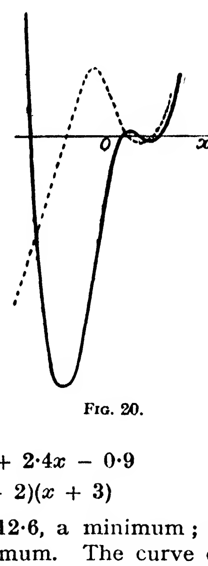


FIG. 20.

Ex. 3.
$$f(x) = 0{\cdot}1x^4 - 1{\cdot}4x^2 + 2{\cdot}4x - 0{\cdot}9$$
$$f'(x) = 0{\cdot}4(x - 1)(x - 2)(x + 3)$$
The stationary values are $f(-3) = -12{\cdot}6$, a minimum; $f(1) = 0{\cdot}2$, a maximum; and $f(2) = -0{\cdot}1$, a minimum. The curve crosses the axis when $x = -4{\cdot}5, 0{\cdot}6, 1{\cdot}6, 2{\cdot}3$ nearly, and is drawn in Fig. 20.

## 52. Rational algebraic functions.

The rational integral algebraic function, or the polynomial of degree $n$, is here denoted by $P_n(x)$; thus, the linear function is $P_1(x)$, the quadratic is $P_2(x)$ and the cubic $P_3(x)$. The quotient of two such polynomials will be denoted by $R(x)$; it is the rational algebraic function, thus
$$R(x) = \frac{P_m(x)}{P_n(x)}$$

We shall suppose that $P_m(x)$ and $P_n(x)$ have no common factor.*

In drawing the graph of $R(x)$, there are certain parts of the curve to which we must pay particular attention, if we wish to obtain a general view of its form, and therefore of the general character of the changes which the function undergoes in its range, or ranges, of definition. These parts are

(1) points at which the graph of $R(x)$ crosses the axis of $x$, which are given by the roots of $P_m(x) = 0$, these are the zeroes of $R(x)$;

(2) points at which the graph of $R(x)$ becomes infinite, which are given by the roots of $P_n(x) = 0$, these are the infinities of $R(x)$;

(3) points at which $R(x)$ has stationary values, given by the roots of $R'(x) = 0$;

(4) the form of the graph as the two ends of the continuum are approached.

**53. Reciprocal of the polynomial function.**

Before discussing in detail the general function $R(x)$, we shall illustrate our methods by the simple though important case of the

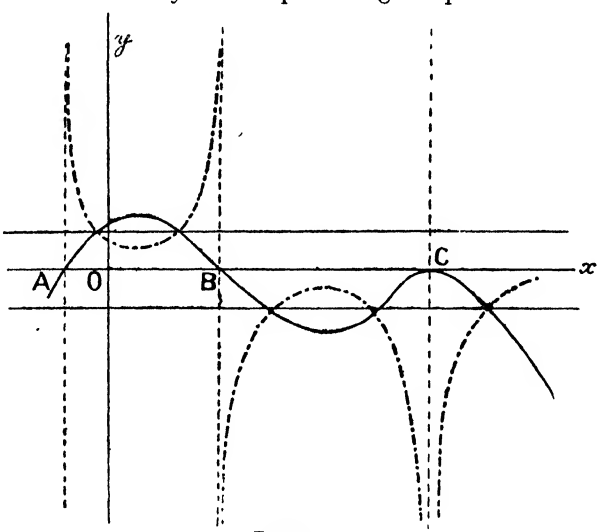



FIG. 21.

reciprocal of the polynomial function. The case arises when $m = 0$; we write
$$R(x) = 1/f(x) \qquad R'(x) = -f'(x)/[f(x)]^2$$

The function $R(x)$ has no zeroes, its infinities correspond to the zeroes of $f(x)$, and its stationary points are given by the same values

* If $P_m(x)$, $P_n(x)$ have a common factor $f(x)$, then $R(x)$ is undefined for values of $x$ which are roots of $f(x) = 0$, but does not differ otherwise from the function obtained by clearing the numerator and denominator of this common factor.

of $x$ as those which give the stationary values of $f(x)$, namely, by the roots of $f'(x) = 0$. Since $R'(x)$ and $f'(x)$ have opposite signs, the maximum stationary values of $f(x)$ correspond to minima of $R(x)$ and the minima of $f(x)$ correspond to maxima of $R(x)$. When $R(x)$ increases, $f(x)$ decreases; indeed, the graph of $R(x)$ is most readily constructed by first drawing the graph of $f(x)$. In the diagram (Fig. 21) $ABC$ is the graph of $f(x)$.

We note that $R(x)$ and $f(x)$ have the same sign, and therefore the graphs of the two functions are both above or both below $Ox$ for the same $x$; also, as $f(x)$ crosses $Ox$ descending as at $B$, $R(x)$ has an infinite discontinuity, changing from $\infty$ to $-\infty$. At $C$, however, $R(x) \rightarrow -\infty$ on both sides of $C$, $f(x)$ vanishing but not becoming positive. To the maximum of $f(x)$, which lies between $A$ and $B$, corresponds a minimum of $R(x)$, and corresponding to the minimum between $B$ and $C$ there is a maximum of $R(x)$. Other important points are those at which the graphs cut, here

$$f(x) = R(x) = 1/f(x)$$

therefore $$f(x) = \pm 1$$

Again, as $x \rightarrow \infty$, and as $x \rightarrow -\infty$, $R(x) \rightarrow 0$, the position of the graph of $R(x)$ being decided by the fact that $R(x)$ and $f(x)$ lie on the same side of $Ox$ for the same value of $x$.

**54. The general type $\quad R(x) = P_m(x)/P_n(x)$**

A few general hints may be given as to methods which are applicable in drawing the graph of $R(x)$, but it is impossible to deal here with all the numerous cases which may arise. The student may learn much from the worked-out examples which follow. The questions raised in Art. 52 must be first dealt with, and the zeroes, infinities and stationary values of the function determined. The discussion of the behaviour of $R(x)$ when $x \rightarrow -\infty$ and when $x \rightarrow \infty$ will be treated more fully by help of the principles laid down in Art. 50. The relative error in taking $ax^m$ for $P_m(x)$ and $a'x^n$ for $P_n(x)$ were shown to be less than any assigned value, if suitably large values of $x$ were taken. If $x$ exceeds the larger of these values, we may study the graph of $R(x)$ at the extremities of its range by examining

$$\frac{ax^m}{a'x^n}$$

The discussion now varies according as $m$ is $>$, $=$ or $< n$; these three cases will be considered separately.

I. If $m < n$, we substitute for $R(x)$, when $x$ is very large, the expression

$$\frac{a}{a'x^{n-m}}$$

Now, this expression is for large values of $x$ a small quantity which, if $n - m$ is even, has the same sign as $a/a'$ whether $x$ is positive or negative. The graph of $R(x)$ in this case approaches $Ox$, lying on the same side of it at both ends.

But, if $n - m$ is odd, $R(x)$ lies on opposite sides of $Ox$ at its extremities, being above the axis when $x \to \infty$ if $a/a'$ is positive, and below if $a/a'$ is negative.

II. If $m = n$, $R(x) \to a/a'$, when $x \to \infty$ and when $x \to -\infty$. Here the position of the line $y = a/a'$ with regard to the curve is similar to that of $Ox$ in Case I. It is usual to express the geometrical fact by saying that $y = a/a'$ is an *asymptote* of the graph. To settle the side upon which the graph lies requires the expansion of the function $R(x)$ in a series of powers of $1/x$. This process is explained below in connection with some examples.

III. If $m > n$, we divide $P_m(x)$ by $P_n(x)$, and write

$$R(x) = f(x) + P_{n-1}(x)/P_n(x)$$

where $f(x)$ is the quotient and $P_{n-1}(x)$ the remainder.

Let $Q_1$ and $Q$ be points on $y = R(x)$, $y = f(x)$ respectively having the same abscissa which is large in magnitude. Then

$$QQ_1 = R(x) - f(x) = P_{n-1}(x)/P_n(x)$$

By the method used above we can take $x$ so great that the right-hand side is, to our degree of approximation,

$$\frac{a_1x^{n-1}}{a'x^n} = \frac{a_1}{a'}\frac{1}{x}$$

where $a_1x^{n-1}$ is the term of highest degree of the remainder.

The signs of $a_1/a'$ and $x$ determine the relative position of the graphs of $R(x)$ and $f(x)$. It is clear that when $x$ is very large the error made in taking $f(x)$ for $R(x)$ is negligible. The curve $y = f(x)$ is an *asymptotic curve*, and when $f(x)$ is linear it is called an *oblique asymptote*. In certain cases it may happen that the remainder is of the $(n - 2)$th degree; in this case, the approximate value of $QQ_1$ is $a_2/(a'x^2)$, a quantity which does not change its sign with $x$; here, $R(x)$ is above or below $f(x)$ at both ends.

## 5. Illustrative examples.

Ex. 1. $\quad R(x) = \dfrac{x - 1}{x(x + 2)} \qquad R'(x) = \dfrac{2 + 2x - x^2}{x^2(x+2)^2}$

The graph crosses $Ox$ at $x = 1$; its infinities are given by $x = 0$, $x = -2$, and its stationary points are the roots of

$$x^2 - 2x - 2 = 0$$

namely, $$x = 1 \pm \sqrt{3}$$

$$R(1 + \sqrt{3}) = \tfrac{1}{2}(2 - \sqrt{3}) = 0{\cdot}13\ldots \quad \text{and} \quad R(1 - \sqrt{3}) = 1{\cdot}87-$$

When $x$ is very large, the value of $R(x)$ may be calculated from its approximation $1/x$; this shows that the graph of $R(x)$ as $x \to -\infty$ and as $x \to \infty$ may be found by drawing the corresponding portions of the rectangular hyperbola $y = 1/x$.

In the range $[-\infty, -2]$ $R(x)$ is negative, in $[-2, 0]$ it is positive, in $[0, 1]$ it is negative and in $[1, \infty]$ it is positive.

The above facts are summarised in the firm lines of the diagram, Fig. 22; the student will find no difficulty in filling in the dotted parts. It is unnecessary to examine in detail these parts except for some particular purpose. The student should aim at acquiring the power of sketching rapidly the graph of such a function as this.

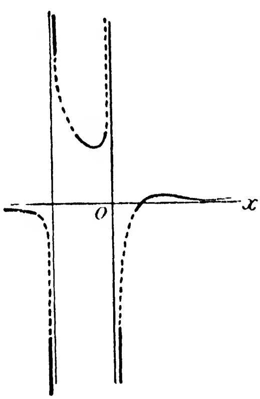



FIG. 22.

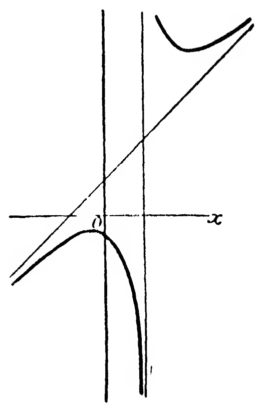



FIG. 23.

Ex. 2. $$R(x) = \frac{x^2 + 1}{x - 2} = x + 2 + \frac{5}{x - 2}$$

$$R'(x) = 1 - \frac{5}{(x - 2)^2}$$

The graph of $R(x)$ is a hyperbola, Fig. 23, which does not cross $Ox$; at $x = 2$, the function is infinite and undefined; its left and right limits as $x$ increases through this value are respectively $-\infty$ and $\infty$. Again

$$y = x + 2$$

is an oblique asymptote. When $x \to \infty$, $R(x)$ is above the asymptote; at the other end, below it. The stationary values are given by $x = 2 \pm \sqrt{5}$, and are equal to $4 \pm 2\sqrt{5}$, that is, to $8{\cdot}47\ldots$, $-0{\cdot}47\ldots$.

Ex. 3. $$R(x) = \frac{x(x - 1)}{(x - 2)(x + 1)} \qquad R'(x) = \frac{2(1 - 2x)}{(x - 2)^2(x + 1)^2}$$

The graph crosses $Ox$ at $x = 0$, $x = 1$; $x = \frac{1}{2}$ gives a maximum value $= \frac{1}{9}$. The lines $x = 2$, $x = -1$ are vertical asymptotes. There is also a horizontal asymptote $y = 1$ deduced from

$$R(x) = 1 + \frac{2}{(x-2)(x+1)} = 1 + \frac{2}{x^2}, \text{ nearly}$$

The asymptote is below the curve at both ends, Fig. 24.

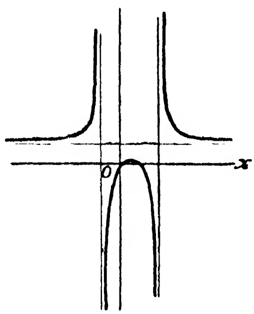



FIG. 24.

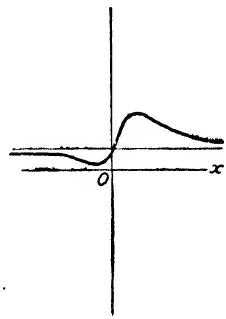



FIG. 25.

Ex. 4. $\quad R(x) = \dfrac{x^2 + x + 1}{x^2 - x + 1} \qquad R'(x) = \dfrac{2(1 - x^2)}{(x^2 - x + 1)^2}$

The curve $y = R(x)$ does not cross $Ox$, and has no vertical or oblique asymptote. The maximum value is given by $R(1) = 3$, and the minimum $R(-1) = \frac{1}{3}$. Again, $x$ being large,

$$R(x) = 1 + \frac{2x}{x^2 - x + 1} = 1 + \frac{2}{x}, \text{ nearly}$$

It follows that $y = 1$ is a horizontal asymptote, which is below the curve when $x \to \infty$ and above it at the other end. It may also be of help to notice that $R(0) = 1$, $R'(0) = 2$, see Fig. 25.

## 56. The function given by $y^2 = R(x)$.

The most general type of algebraic function is given when $x$ and $y$ are united by the relation

$$y^n f_0(x) + y^{n-1} f_1(x) + \dots + y f_{n-1}(x) + f_n(x) = 0$$

where $f_0(x), f_1(x) \dots f_n(x)$ are polynomials of any degree. Functions of this kind will occur often at different stages of the subject, but at present the case which we select for discussion is the algebraic function, which is defined by

$$y^2 = R(x) = P_m(x)/P_n(x)$$

We first draw the graph of $R(x)$, and then notice that $y$ is not defined when $R(x)$ is negative, while when $R(x)$ is positive there are two

values of $y$ equal in magnitude and opposite in sign. Other important, though simple, facts are expressed analytically by

$$R(x) = 0 \qquad y = 0; \qquad R(x) \rightarrow +\infty \qquad y \rightarrow \pm\infty$$

An interesting feature of the graph is the point at which $R(x) = 0$; three typical cases are expressed by the equations

$$y^2 = (x - a)R_1(x) \qquad y^2 = (x - a)^2R_1(x) \qquad y^2 = (x - a)^3R_1(x)$$

where $R_1(a) \neq 0$. In the first case the student may prove that the gradient at $x = a$ is infinite; that in the second it is $\pm\sqrt{R_1(a)}$; that in the third it is zero. For further illustration the reader is referred to Ex. 1, 2, 3 which follow.

**57. Illustrative examples.**

Ex. 1. $$y^2 = x(x - 1)(x - 2) \quad \text{(Fig. 26)}$$

Here, in the ranges $[-\infty, 0]$ and $[1, 2]$, $y$ is undefined: in $(0, 1)$ the values of $y$ give an oval, while in $(2, \infty]$ we have a symmetrical branch extending to infinity. The infinite branch is best realised by considering the semicubical parabola $y^2 = x^3$ to which the curve approximates when $x$ is large.

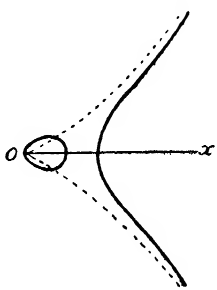



FIG. 26.

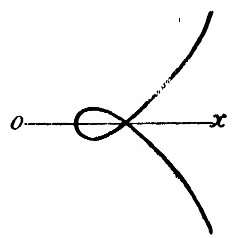



FIG. 27.

Ex. 2. $$y^2 = (x - 1)(x - 2)^2 \quad \text{(Fig. 27)}$$

Here, when $x < 1$, $y$ is undefined. Also

$$\pm 2\frac{dy}{dx} = \frac{x - 2}{\sqrt{(x - 1)}} + 2\sqrt{(x - 1)} \,*$$

Hence, at $(2, 0)$ the gradient is equal to $\pm 1$, and the curve crosses itself at this point. At $x = 1\frac{1}{3}$, $y$ has stationary values.

Ex. 3. $$y^2 = (x - 2)^3 \quad \text{(Fig. 28)}$$

In this curve, $y$ is defined only when $x \geq 2$; the point $(2, 0)$ is a cusp, the branches touching $Ox$ at this point, for

$$\frac{dy}{dx} = \frac{3}{2}(x - 2)^{\frac{1}{2}}$$

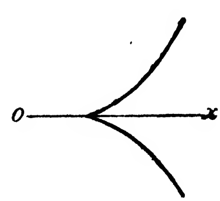



FIG. 28.

* This is an anticipation of results proved in Chapter VI.

Ex. 4. $$y^2 = \frac{x+1}{x+2} \quad \text{(Fig. 29)}$$

A consideration of the function $(x + 1)/(x + 2)$ shows that when $x \to \infty$, $y^2 \to 1$, that $y^2$ is positive, except in the range $(-2, -1)$. The curve required has two horizontal asymptotes $y = \pm 1$, and a vertical asymptote $x = -2$.

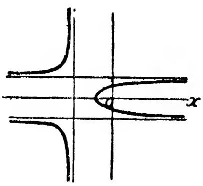


FIG. 29.

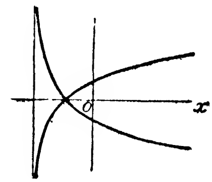


FIG. 30.

Ex. 5. $$y^2 = \frac{(x+1)^2}{x+2} \quad \text{(Fig. 30)}$$

The graph of $(x + 1)^2/(x + 2)$ is a hyperbola whose asymptotes are $x = -2$, $y = x$, and which touches $Ox$ at $x = -1$; also this hyperbola is above $Ox$ from $[-2, \infty]$. The curve whose equation is given may be drawn from these facts about the hyperbola. It can be shown that the gradient of the given curve at $(-1, 0)$ is $\pm 1$. Its infinite branches, which are in the first and fourth quadrants, are given by the approximation $y^2 = x$.

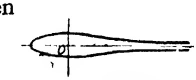


FIG. 31.

Ex. 6. $$y^2 = \frac{x+1}{(x+2)^2} \quad \text{(Fig. 31)}$$

The range of values of $x$ for which $y$ is defined is $(-1, \infty]$, stationary values are given by $x = 0$, for we have, as will be proved in Chapter VI.,

$$2y\frac{dy}{dx} = -\frac{x}{(x+2)^3}$$

## EXERCISES V.

1. Express the following quadratic functions as the sum, or the difference, of the square of a linear function and a constant

i. $3x^2 - 6x + 1$ ii. $(x + 3)^2 + (x - 5)^2$
iii. $2(2x + 1)^2 - (x - 3)^2$ iv. $x^2 - (x + 1)^2 - (x + 2)^2$
v. $x^2 - (x + 1)^2 + (x + 2)^2$ vi. $x^2 + (x + 1)^2 - (x + 2)^2$

2. Trace the graphs of the following cubic functions

i. $x^3 - 3x + 1$ ii. $x^3 - 3x + 6$ iii. $x^3 + 6x - 9$
iv. $x^3 + 6x + 1$ v. $x^3 + 3x^2 - 9x$ vi. $x^3 + 3x^2 - 9x + 6$
vii. $x^3 + 3x^2 - 9x - 28$ viii. $x^3 + 2x^2 - 5x$ ix. $x^3 + x^2 - 5x + 3$

3. Trace the graphs of the following biquadratic (or quartic functions

i. $x^4 - 2x^2 - 2$ ii. $x^4 - 2x^2 + 3$
iii. $x^4 - 4x^3 + 4x^2 + 1$ iv. $x^4 - 4x^3 + 4x^2 - 1$
v. $3x^4 + 4x^3 + 6x^2 - 2$ vi. $3x^4 + 4x^3 + 6x^2 + 2$
vii. $x^4 - 6x^2 + 8x - 3$ viii. $x^4 - 6x^2 + 8x + 25$

4. Prove that $x^4 + ax^2 + b$ can always be resolved into the product of two quadratic factors with real coefficients.

Resolve into real quadratic factors

i. $x^4 + x^2 + 1$ ii. $x^4 - 3x^2 + 1$ iii. $x^4 + 9$

5. Show that $x^4 - 4x + 10 = 0$ has no real roots, and in general that $x^4 - 4p^3x + q = 0$ has no real roots, if $q > 3p^4$.

6. If $f(x) = 0$ has a single root in the neighbourhood of $x = a$, then

$$a_1 = a - f(a)/f'(a)$$

is nearer the root than $a$, if for all values of $x$ in $[\frac{1}{2}(a + a_1), a]$ the relation $|(f'(x)| \leq 2|f'(a)|$ holds.

*Note.*—If the tangent at the point $(a, f(a))$ of the graph of $f(x)$ meets the axis of $x$ at $T$ and if $NP$ is the ordinate of $P$ and $A$ bisects $TN$, then $x = OT$ is a better approximation to the root of $f(x) = 0$ than $x = ON$, provided the point $Q$, where the graph cuts $Ox$, is in $TA$; $Q$ is certainly in $TA$, if the gradient of the graph at all points above $AN$ is less than the gradient of $AP$.

7. Find approximations to the roots of the following cubic equations, using Question 6,

i. $x^3 - 9x - 14 = 0$ ii. $x^3 - 12x + 20 = 0$
iii. $x^3 + 3x^2 - 24x - 16 = 0$

8. Prove that the real roots of $x^4 - 12x - 5 = 0$ are approximately $2{\cdot}41\ldots$, $-0{\cdot}41\ldots$.

9. Draw the graphs of the following functions, determining any stationary values they may have,

i. $\dfrac{x^2}{x^2 + x + 1}$ ii. $\dfrac{x^2 - 1}{x^2 - 4}$ iii. $\dfrac{x(x - 4)}{x^2 + 9}$

iv. $\dfrac{x^2 + 9}{x(x - 4)}$ v. $\dfrac{x^2 + 8x + 16}{x^2 - 6x + 9}$ vi. $\dfrac{x^2 + 2x - 6}{x + 5}$

vii. $\dfrac{x^2 + 9}{x^2 + 2x + 9}$ viii. $\dfrac{(x + 1)(x - 3)}{(x - 4)(x - 5)}$ ix. $\dfrac{x(x - 2)}{(x - 1)(x - 3)}$

x. $\dfrac{(x - 1)(x - 5)}{(x - 2)^2}$ xi. $\dfrac{(ax + b)^2}{x}$ xii. $\dfrac{x^2 + 2x + 3}{3x^2 + 2x + 1}$

10. Show that the stationary values of

$$\frac{x^2 + a^2}{(x - c)(x - d)}$$

are given by rational values of $x$, if $(a^2 + c^2)(a^2 + d^2)$ is a perfect square.

11. Prove that
$$\frac{(ax - b)(bx + a)}{(bx - a)(ax + b)}$$
has all values.

12. Show that
$$\frac{ax^2 + bx + c}{cx^2 + bx + a}$$
has no stationary values if $b^2 > (a + c)^2$. What is the type of the graph of the function when $b^2 < ac$ ?

13. Sketch the curves represented by the following equations

i. $y^2 = x^2(x^2 - 4)$  ii. $y^2 = \dfrac{x}{x - 2}$  iii. $y^2 = \dfrac{x^2}{x - 2}$

iv. $y^2 = \dfrac{x + 2}{x^2}$  v. $y^2 = x^2\dfrac{x - 1}{x + 1}$  vi. $y^2 = \dfrac{x^2 + 1}{x^2 - 1}$.

vii. $y^2 = \dfrac{x^2 + 1}{x}$  viii. $y^2 = \dfrac{x}{x^2 + 1}$  ix. $y^2 = \dfrac{x^2 - 1}{x}$

x. $y^2 = \dfrac{x}{x^2 - 1}$  xi. $y^2 = x^2(x - 1)(x - 2)$

xii. $y^2 = x^3(x - 2)$  xiii. $y^2 = x^2\dfrac{1 - x^2}{1 + x^2}$  xiv. $y^2 = x^2\dfrac{1 + x^2}{1 - x^2}$

# CHAPTER VI

## INVERSE OF A FUNCTION, FUNCTION OF A FUNCTION

**58. The square root function.**

There are certain mathematical operations which are called direct, such as addition, multiplication and involution. We have been concerned hitherto mainly with functions which arise from the use of such operations, types of which are

$$x + 2 \qquad 3x \qquad x^2$$

But to each of the three direct operations mentioned above, an inverse operation corresponds. Passing over the operations of subtraction and division, which are respectively the inverses of addition and multiplication, we base the explanation of inverse functions upon the inverse function associated with the simple quadratic function $x^2$. We have used the table

| $x$ | ... | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| $x^2$ | ... | 9 | 4 | 1 | 0 | 1 | 4 | 9 |

to illustrate the part played by a mathematical operation in the construction of a function. It is necessary to remind the student that the selection of the values of $x$ is purely arbitrary and is made from a vast assemblage of numbers, rational and irrational, any of which might have been taken—the whole of which constitutes the arithmetical continuum. The selection made is a concession to human weakness and its preference for the integral elements of the continuum. Thus, the table might have been written

| $x$ | $-2$ | $-1{\cdot}732$ | $-1{\cdot}414$ | $-1$ | 0 | 1 | $1{\cdot}414$ | $1{\cdot}732$ | 2 | $2{\cdot}236$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $x^2$ | 4 | $3{\cdot}00$ | $2{\cdot}00$ | 1 | 0 | 1 | $2{\cdot}00$ | $3{\cdot}00$ | 4 | $5{\cdot}00$ |

In this form the selection of values of the independent variable gives values of the dependent variable which to two places of

decimals are integers. Now suppose that we interchange the first and second row, calling the new upper row $x$ and the new lower row $f(x)$; we then have the table of a new function, which may be written

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0 | $\pm 1$ | $\pm 1{\cdot}414\ldots$ | $\pm 1{\cdot}732\ldots$ | $\pm 2$ | $\pm 2{\cdot}236\ldots$ |

The new function is closely related to the old function of squares. What label can we apply to the new function? The answer is supplied by our knowledge of arithmetic; the new function is the square root function, the function which we denote by the symbols $\pm\sqrt{x}$ or $x^{\frac{1}{2}}$; the numerical magnitudes in the lower row being derived from the corresponding terms of the upper row by taking the square root.

**59. General inverse functions.**

The process which we have followed in deducing the function $x^{\frac{1}{2}}$ from the function $x^2$ is perfectly general. For the correspondence between two sets of numbers, which we term the independent ($x$) and the dependent variable ($y$), implies also a correspondence between the $y$'s and the $x$'s, in which $y$ plays the part of the independent variable. *Every function is therefore associated with a second function called its inverse, the same pairs of values being coupled in the tables of the functions.*

**60. Some properties of inverse functions.**

The polynomial functions are single-valued, that is, to each value of $x$ a single value of $y$ corresponds; also their range is unrestricted, that is, whatever number is assigned to $x$ in the arithmetical continuum, a corresponding value of $y$ can be found. But the inverse functions of single-valued functions of unrestricted range need not be single-valued, and may be restricted in range. In the case selected $x^{\frac{1}{2}}$ is a function which has two values for every positive value of $x$, no value when $x$ is negative and a single value when $x = 0$.

The student should form by the help of tables the functions which are the inverses of

$$x + 2 \qquad 3x \qquad x^3$$

and he will find that he obtains the functions

$$x - 2 \qquad \tfrac{1}{3}x \qquad \sqrt[3]{x}$$

These three functions are, like the original function, single-valued and of unrestricted range.

**61. The graph of an inverse function**

The connection between the graphs of a function and of its inverse can be studied best by drawing the two graphs upon one diagram.

Let $(x, y)$ be $P$, a point on the graph of the function, and $(X, Y)$ be $P'$, the corresponding point on the graph of the inverse function. Then

$$X = y \qquad Y = x$$

It is obvious (see Fig. 32) that the perpendicular from $O$ upon $PP'$ bisects $PP'$ and also bisects the angle $xOy$. Then $P$, $P'$ are the images of each other in the diagonal line $y = x$ of the square paper which passes through $O$, and this is true for each pair of corresponding points. It follows that *the graph of a function and of the inverse function are the images of each other in the diagonal line* $y = x$.

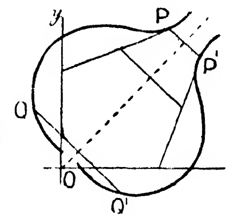


FIG. 32.

It follows from either the arithmetical or the geometrical discussion that *the inverse of the inverse of a function is the original function.*

Certain functions are their own inverses, *e.g.* the functions $-x$, $1/x$. The graphs of such functions are symmetrical with respect to $y = x$.

**62. Differential coefficient of an inverse function.**

It is required *to prove that*

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$$

We begin with $y = f(x)$, and represent the inverse relation between the same pair of variables by $x = \varphi(y)$, or, if in the second relation we put $x = Y$, $y = X$, we have $Y = \varphi(X)$. The relation which we now have to establish takes the form

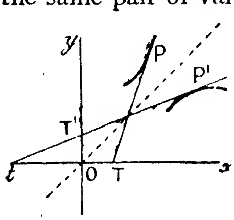


FIG. 33.

$$f'(x) \cdot \varphi'(X) = 1,$$

and will be proved first from the geometrical properties of the graphs of $f(x)$ and $\varphi(X)$.

The tangents at $P$ and $P'$ being symmetrical with respect to the diagonal line through $O$ must meet on it, also

$$PTx = P'T'y = 90° - P'tx$$

provided neither tangent is parallel to an axis. It follows that

$$\tan PTx = \cot P'tx$$
$$f'(x) = 1/\varphi'(X)$$
$$f'(x) \cdot \varphi'(X) = 1$$

that is,

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$$

This method of proof brings out the fact that in $dy/dx$ * $y$ is thought of as a function of $x$, while in $dx/dy$ $x$ is a function of $y$, namely, that function which is the inverse of the first function considered.

In the following analytical proof this important consideration may be easily overlooked.

Let $x$, $y$ be a pair of corresponding values of the function ; $y$, $x$ are the corresponding pair of the inverse function. Further, $x + \delta x$, $y + \delta y$ is a consecutive pair of corresponding values ; then, since $\delta x$, $\delta y$ are both finite,

$$\frac{\delta y}{\delta x} \cdot \frac{\delta x}{\delta y} = 1$$

Now as $\delta x \to 0$, $\delta y \to 0$, and

as $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$ (where $x$ is independent variable)

as $\delta y \to 0$, $\frac{\delta x}{\delta y} \to \frac{dx}{dy}$ (where $y$ is independent variable)

With the assumption that neither of these limiting values vanishes, we have the theorem to be proved

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$$

**63. The differential coefficient of $\sqrt[q]{x}$.**

If $y = \sqrt[q]{x} = x^{1/q}$, where $q$ is an integer, then the inverse function is given by the relation $\quad x = y^q$

This function we can differentiate by the rules laid down in Art. 37, and we have

$$\frac{dx}{dy} = qy^{q-1} = qx^{(q-1)/q}$$

Hence $\quad \frac{dy}{dx} = 1 \bigg/ \frac{dx}{dy} = \frac{1}{q} x^{-(q-1)/q} = \frac{1}{q} x^{1/q - 1}$

**64. The differential coefficient of $x^{p/q}$.**

First, we take $p$, $q$ as positive integers. Then

$$y = x^{p/q} = \sqrt[q]{x} \, . \sqrt[q]{x} \ldots (p \text{ factors})$$

$$\frac{dy}{dx} = (\sqrt[q]{x})^{p-1} \frac{d}{dx} \sqrt[q]{x} + \ldots + \ldots (p \text{ terms})$$

$$= px^{(p-1)/q} \, . \frac{1}{q} x^{1/q - 1} = \frac{p}{q} x^{p/q - 1}.$$

* This notation for the differential coefficient is used very sparingly in this book and then only for typographical reasons. It does not imply that the differential coefficient is a fraction with a numerator and a denominator.

Secondly, we take the case in which

$$y = x^{-p/q} = 1/x^{p/q}$$

$$\frac{dy}{dx} = \frac{0 \,.\, x^{p/q} - \dfrac{d}{dx} x^{p/q}}{x^{2p/q}} = -\frac{d}{dx} x^{p/q} \div x^{2p/q}$$

$$= -\frac{p}{q} x^{p/q - 1} \div x^{2p/q} = -\frac{p}{q} x^{-p/q - 1}$$

The results proved allow us now to assert that for all commensurable values of $n$,

$$\frac{d}{dx} x^n = nx^{n-1}$$

The discussion of incommensurable numbers which has been given may perhaps prepare the student to accept the extension of this formula to the case of $n$ being an incommensurable or irrational number, the knowledge which he possesses is hardly sufficient to justify us in embarking upon a formal proof of the theorem.

**65. Function of a function, compound function.**

The function $\sqrt{(x^2 + 1)}$ is chosen to illustrate the class of functions considered under this heading. The construction of such a function depends upon two tables, a table of squares and a table of square roots. The function is the square root of a quadratic function of $x$; its structure may be exhibited by writing down the following sequence of functions, each of which specifies a stage in the construction

$$x$$
$$x^2$$
$$x^2 + 1$$
$$\sqrt{(x^2 + 1)}$$

Two intermediate stages are shown; the process might have been shortened by the omission of the first stage, in which case we should pass from $x$ to the quadratic function $x^2 + 1$. This process is expressed by an analysis in which $z$ is substituted for the intermediate function. Thus

$$z = x^2 + 1 \qquad y = \sqrt{(x^2 + 1)} = \sqrt{z}$$

The reader may notice that the importance of this analysis for us lies in the fact that we have resolved the function into two, each of which can be differentiated by the rules already laid down. Thus

$$\frac{dz}{dx} = 2x \qquad \frac{dy}{dz} = \tfrac{1}{2} z^{-\frac{1}{2}}$$

It is with this object that we break up the compound function into its simpler elements, the intermediate functions used being those whose differential coefficients we can form.

**66. The differential coefficient of $\mathbf{f[\phi(x)]}$.**

Let us write $y = f(z)$, $z = \varphi(x)$, where we suppose that $f'(z)$, $\varphi'(x)$ are known, and by their nature are finite. Let $x$ be changed into $x + \delta x$, and let the value of $z$ (which is paired with $x$ in the function $z = \varphi(x)$) become $z + \delta z$; further, let $y + \delta y$ correspond to the value $z + \delta z$. Then we have

$$y + \delta y = f(z + \delta z) \qquad z + \delta z = \varphi(x + \delta x).$$

Now, since $\delta x$, $\delta y$, $\delta z$ are all finite increments,

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x}$$

And as $\delta x \to 0$, we have $\delta z \to 0$, and consequently $\delta y \to 0$.

Again, as $\delta x \to 0$,

$$\frac{\delta y}{\delta x} \to \frac{dy}{dx} \qquad \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x} \to \frac{dy}{dz} \cdot \frac{dz}{dx}$$

and since in all stages of the sequences which these limits imply

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x}$$

the limiting values of the two sides are equal, that is,

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

The same result may be written by means of the functional symbols

$$y = f[\varphi(x)]$$

in the form
$$\frac{dy}{dx} = f'[\varphi(x)] \cdot \varphi'(x)$$

Taking the example given in Art. 65,

$$y = \sqrt{(x^2 + 1)} = z^{\frac{1}{2}} \qquad z = x^2 + 1$$

$$\frac{dy}{dz} = \frac{1}{2}z^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{(x^2 + 1)}} \qquad \frac{dz}{dx} = 2x$$

Therefore
$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{2}\frac{1}{\sqrt{(x^2 + 1)}}2x = \frac{x}{\sqrt{(x^2 + 1)}}$$

The method given above is of great importance, as it allows us to increase very considerably the stock of functions which we may differentiate. In some of the examples worked out below, alternative methods of differentiation are employed in order to bring the

newer method into relation with the processes explained and used at an earlier stage, but it is not always possible to give alternative solutions.

**67. Illustrative examples.**

Ex. 1. $$y = (1 - x)^3 = 1 - 3x + 3x^2 - x^3$$

$$\frac{dy}{dx} = -3 + 6x - 3x^2 = -3(1 - x)^2$$

Again, $$\frac{dy}{dx} = \frac{d(1 - x)^3}{d(1 - x)} \cdot \frac{d(1 - x)}{dx} = 3(1 - x)^2(-1)$$

$$= -3(1 - x)^2$$

The same result is obtained; the second method may also be carried out with the substitution $z = (1 - x)$.

Ex. 2. $$y = \left(\frac{1 - x}{1 + x}\right)^2 = \frac{1 - 2x + x^2}{1 + 2x + x^2}$$

$$\frac{dy}{dx} = \frac{(-2 + 2x)(1 + x)^2 - (2 + 2x)(1 - x)^2}{(1 + x)^4}$$

$$= -\frac{4(1 - x)}{(1 + x)^3}$$

Also, writing $z = (1 - x)/(1 + x)$, $y = z^2$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 2z \cdot \frac{-(1 + x) - (1 - x)}{(1 + x)^2} = 2z \cdot \frac{-2}{(1 + x)^2}$$

$$= \frac{-4(1 - x)}{(1 + x)^3}$$

Ex. 3. $$y = \sqrt{\left(\frac{1 + x}{1 - x}\right)}$$

Here $$y = z^{\frac{1}{2}} \qquad z = \frac{1 + x}{1 - x}$$

$$\frac{dy}{dz} = \frac{1}{2}z^{-\frac{1}{2}} = \frac{1}{2}\sqrt{\left(\frac{1 - x}{1 + x}\right)} \qquad \frac{dz}{dx} = \frac{2}{(1 - x)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{(1 - x)\sqrt{(1 - x^2)}}$$

Ex. 4. $$y = \frac{x}{\sqrt{(1 - x^2)}}$$

Here we have to find the differential coefficient of $\sqrt{(1 - x^2)}$ with regard to $x$; we may arrange the work thus

$$\frac{dy}{dx} = \frac{\sqrt{(1 - x^2)} - x\dfrac{d}{dx}\sqrt{(1 - x^2)}}{1 - x^2} = \frac{(1 - x^2)^{\frac{1}{2}} - x\frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)}{1 - x^2}$$

$$= \frac{1 - x^2 + x^2}{(1 - x^2)^{\frac{3}{2}}} = \frac{1}{(1 - x^2)^{\frac{3}{2}}}$$

### EXERCISES VI (A). Inverse Functions.

1. Show that the following pairs of functions $f(x)$, $\varphi(x)$ are such that $f[\varphi(x)] = x$, $\varphi[f(x)] = x$.

i. $f(x) = \dfrac{x+1}{x-2}$ $\qquad \varphi(x) = \dfrac{2x+1}{x-1}$

ii. $f(x) = \dfrac{3x}{x-1}$ $\qquad \varphi(x) = \dfrac{x}{x-3}$

iii. $f(x) = \dfrac{x^2}{2x-1}$ $\qquad \varphi(x) = x \pm \sqrt{(x^2 - x)}$

iv. $f(x) = \left(\dfrac{x}{x-4}\right)^{\frac{1}{2}}$ $\qquad \varphi(x) = \dfrac{4x^2}{x^2-1}$

Draw the graphs of $f(x)$ and $\varphi(x)$.

2. Prove that $y = 1 - x^3$ cuts the axis of $x$ at an angle equal to the angle at which $x = 1 - y^3$ cuts $Oy$.

3. Verify that (i) the gradient of $y(x^2 + 1) = 6$ at $(1, 3)$ is the reciprocal of the gradient of $y^2x = 6 - x$ at $(3, 1)$, (ii) the gradient of $y(x - 1) = 3x$ at $(2, 6)$ is equal to the reciprocal of the gradient of $(x - 3)y = x$ at $(6, 2)$, and (iii) the gradient of $y(2x - 1) = x^2$ at $(2 + \sqrt{2}, 2)$ is the reciprocal of the gradient of $y = x \pm \sqrt{(x^2 - x)}$ at $(2, 2 + \sqrt{2})$.

4. If the parabolas

$$y = ax^2 + 2bx + c \qquad x = ay^2 + 2by + c$$

touch, show that $\qquad (2b - 1)^2 = 4ac$ or $4(ac + 1)$.

5. Show that if the graph of $f(x)$, traced upon transparent paper, is viewed from the other side of the paper, and the axes of $x$ and $y$ interchanged, the resulting diagram is the graph of the inverse of $f(x)$.

### EXERCISES VI (B). Differentiation.

Differentiate the following functions

| | | |
|---|---|---|
| 1. $(2x + 1)^2$ | 2. $(1 - 2x)^2$ | 3. $(2 - 3x)^3$ |
| 4. $\sqrt{(2 - 3x)^3}$ | 5. $(x + 1/x)^2$ | 6. $\sqrt{(x - 1/x)}$ |
| 7. $\dfrac{(1 + 2x)^3}{(1 + 3x)^2}$ | 8. $\dfrac{(1 - 3x)^2}{(1 - 2x)^3}$ | 9. $\sqrt{\left(\dfrac{1 - x}{1 + x}\right)}$ |
| 10. $\sqrt{\left(\dfrac{1 + x^2}{1 - x^2}\right)}$ | 11. $\dfrac{x}{\sqrt{(a^2 - x^2)}}$ | 12. $\dfrac{x}{\sqrt{(a^2 + x^2)}}$ |
| 13. $(1 + x)^3(1 - x)^2$ | 14. $(1 - x)^2(2x + 3)^3$ | 15. $(x + a)^p(x + b)^q$ |
| 16. $\dfrac{\sqrt{(a^2 + x^2)}}{x}$ | 17. $\dfrac{\sqrt{(a^2 - x^2)}}{x}$ | 18. $\sqrt{\left(\dfrac{a^2 - x^2}{a^2 + x^2}\right)}$ |

19. $\left(\frac{a + x}{a - x}\right)^n$ 20. $\frac{a + x}{\sqrt{(a^2 + x^2)}}$ 21. $\frac{x^2 - ax}{\sqrt{(x^2 + a^2)}}$

22. $(1 + x)^n + (1 - x)^n$ 23. $(1 + x^n)^n + (1 - x^n)^n$

24. $\frac{\sqrt{(a + x)} + \sqrt{(a - x)}}{\sqrt{(a + x)} - \sqrt{(a - x)}}$ 25. $(a + 2bx + cx^2)^n$

26. $(4x - 7)(3x + 7)^{\frac{4}{3}}$ 27. $(8x^4 + 4x^2 + 3)\sqrt{(x^2 - 1)}/x^5$

28. $(1 + x^2)\sqrt{(1 - x^2)}/x$ 29. $(10 - 6x + 3x^2)(5 + 2x)^{\frac{3}{2}}$

30. $(2x^2 + 3)(3 - 5x^2)^{\frac{5}{2}}/x^7$ 31. $\sin^2 x$

32. $\sin^2 2x$ 33. $(\sin \frac{1}{2}x)^{\frac{1}{2}}$ 34. $\cos 2x$

35. $\sqrt{\cos 2x}$ 36. $\sec^2 2x$ 37. $\sec^2 x - \tan^2 x$

38. $\sin^2 x - \cos^2 x$ 39. $4 \cos^3 x - 3 \cos x$ 40. $\tan^3 3x$

41. $3 \tan x + \tan^3 x$ 42. $\sec^3 x - 3 \sec x$ 43. $\cos^2 x/x^2$

44. $x^2 \operatorname{cosec}^2 x$ 45. $\tan^2 2(x + \alpha)$ 46. $\frac{\sin x}{4 \cos^2 x - 3}$

47. $\sqrt{\left(\frac{1 - \sin x}{1 + \sin x}\right)}$ 48. $\frac{4 \cos^2 x - 3}{\sin x}$ 49. $\frac{1}{2}x - \frac{1}{4} \sin 2x$

50. $\frac{1}{3} \cos^3 x - \cos x$ 51. $\frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x$

52. $x + \cot x$ 53. $\tan x - x$ 54. $\frac{1}{3} \tan^3 x - \tan x + x$

55. Differentiate $x^5$ with respect to $x^3$ and $x^p$ with respect to $x^q$.

56. Differentiate $(ax + b)/(cx + d)$ with respect to $(a'x + b')/(c'x + d')$.

57. Differentiate $1 + x^2$ with respect to $1 - x^2$.

58. Differentiate $\sin x$ with respect to $\cos x$ and $\sin 3x$ with respect to $\cos 2x$.

59. Differentiate $\sin^2 x$ with respect to $\cos^3 x$.

60. Differentiate $\tan x$ with respect to $\sec x$ and $\sec x$ with respect to $\tan x$.

# CHAPTER VII

## TANGENT AND NORMAL

**68. Equations of the tangent and normal of $\mathbf{y = f(x)}$.**

It has been proved in Art. 33 that the gradient at any point on the graph of $f(x)$ is equal to the differential coefficient of the function at the point considered. We may define the tangent at $P$ as the straight line through $P$ which has the same gradient as the curve at this point. Let $P$ be the point $(x_1, y_1)$, and let the equation of the tangent be

$$y = ax + b \qquad \qquad (1)$$

Then the gradient of (1) is equal to $f'(x_1)$; therefore

$$a = f'(x_1) \qquad \qquad (2)$$

Again, since $(x_1, y_1)$ lies upon the tangent,

$$y_1 = ax_1 + b \qquad \qquad (3)$$

Subtracting (3) from (1)

$$y - y_1 = a(x - x_1)$$

and the value of $a$ from (2) gives

$$y - y_1 = f'(x_1)(x - x_1)$$

as the equation of the tangent at $(x_1, y_1)$.

Again, the normal is a straight line through $P$ perpendicular to the tangent at $P$. The equation of the normal at $P$ is therefore

$$y - y_1 = -(x - x_1)/f'(x_1)$$

**69. Illustrative examples.**

Ex. 1. *To find the tangent and normal at $(x_1, y_1)$ of the curve $y = x^2/4a$.*

Now $\qquad f(x) = x^2/4a \qquad f'(x) = x/2a$

The equation of the tangent is

$$\begin{aligned} y - y_1 &= x_1(x - x_1)/2a \\ 2ay - 2ay_1 &= xx_1 - x_1^2 \\ &= xx_1 - 4ay_1 \end{aligned}$$

that is, $\qquad 2a(y + y_1) = xx_1$

The equation of the normal is

$$y - y_1 = -2a(x - x_1)/x_1$$

which may be modified by writing $y_1 = x_1^2/4a$.

Ex. 2. *To find the tangent and normal of $y = x^3 + 3x^2$ at the point whose abscissa is* 1.

Now $f(x) = x^3 + 3x^2 \quad f'(x) = 3x^2 + 6x$

Writing $x = 1 \quad f(1) = 4 \quad f'(1) = 9$

The equation of the tangent is

$$y - 4 = 9(x - 1)$$
$$y = 9x - 5$$

The equation of the normal is

$$y - 4 = -\tfrac{1}{9}(x - 1)$$
$$x + 9y = 37.$$

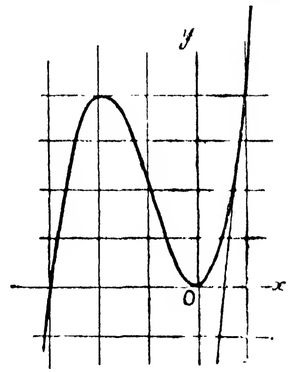



FIG. 34.

In an example such as this, which admits of representation on squared paper, the student should draw the graphs of the function and the tangent on a largish scale, and convince himself so far as the test may allow, that the line touches the curve ; see Fig. 34.

**70. The algebraic equation of a curve.**

It is often difficult, and sometimes impossible, to write the relation between the coordinates of a point on a curve in the form $y = f(x)$, where $f(x)$ is a simple function. Even the circle whose equation, $x^2 + y^2 - a^2 = 0$, may be written $y = \pm\sqrt{(a^2 - x^2)}$ is dealt with more conveniently in the first form, and this is certainly true of the general equation of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Generally, we shall write the equation of a curve in the notation

$$f(x, y) = 0$$

where, in Section 1 of this book, we mean by $f(x, y)$ a sum of a finite number of products of powers of $x$ and $y$ ; thus

$$f(x, y) = \mathrm{A}x^m y^n + \mathrm{B}x^p y^q + \dots + \mathrm{H}x^u y^v$$

Now, if we write $\qquad z = f(x, y)$

we have

$$\frac{dz}{dx} = \frac{d}{dx}(\mathrm{A}x^m y^n) + \frac{d}{dx}(\mathrm{B}x^p y^q) + \dots$$

$$= \mathrm{A}mx^{m-1}y^n + \mathrm{A}x^m\frac{d}{dx}y^n + \mathrm{B}px^{p-1}y^q + \mathrm{B}x^p\frac{d}{dx}y^q + \dots$$

Here $y^n$, $y^q$, ..., being powers of a function of $x$, may be differentiated by the rule explained in Art. 66, that is,

$$\frac{d}{dx}y^m = my^{m-1}\frac{dy}{dx} \qquad \frac{d}{dx}y^q = qy^{q-1}\frac{dy}{dx}$$

It follows that

$$\frac{dz}{dx} = \mathrm{A}x^{m-1}y^{n-1}\left(my + nx\frac{dy}{dx}\right) + \mathrm{B}x^{p-1}y^{q-1}\left(py + qx\frac{dy}{dx}\right) + \dots$$

Again, since for all variations of $x$ and $y$ which satisfy

$$f(x, y) = 0$$

we have $z = 0$, we must also have $dz/dx = 0$; it follows that

$$\mathrm{A}x^{m-1}y^{n-1}\left(my + nx\frac{dy}{dx}\right) + \mathrm{B}x^{p-1}y^{q-1}\left(py + qx\frac{dy}{dx}\right) + \dots = 0$$

an equation which determines the gradient of the curve at $(x, y)$.

**71. Illustrative examples.**

Ex. 1. *To find the tangent of* $x^2 + y^2 = a^2$ *at* $(x_1, y_1)$.

Now, on differentiation we have

$$\frac{d}{dx}(x^2 + y^2 - a^2) = 0$$

that is,
$$2x + 2y\frac{dy}{dx} = 0$$

Therefore
$$\frac{dy}{dx} = -\frac{x}{y}$$

The equation of the tangent is

$$y - y_1 = -x_1(x - x_1)/y_1$$

that is,
$$xx_1 + yy_1 = x_1^2 + y_1^2 = a^2$$

Ex. 2. *To find the tangent of* $y^2 = 4ax$ *at* $(x_1, y_1)$.

By differentiating $y^2 = 4ax$, we obtain

$$2y\frac{dy}{dx} = 4a$$

The gradient at $(x_1, y_1)$ is $2a/y_1$, and the equation of the tangent is

$$y - y_1 = 2a(x - x_1)/y_1$$
$$yy_1 - 4ax_1 = 2ax - 2ax_1$$
$$yy_1 = 2a(x + x_1)$$

Ex. 3. *To find the tangent of* $2xy = c^2$ *at* $(x_1, y_1)$.

Differentiating $2xy = c^2$, we have

$$\frac{d}{dx}(xy) = 0$$

$$x\frac{dy}{dx} + y = 0$$

The gradient at $(x_1, y_1)$ is $-\ y_1/x_1$, and the equation of the tangent is

$$y - y_1 = -\ y_1(x - x_1)/x_1$$
$$xy_1 + yx_1 = 2x_1y_1 = c^2$$

Ex. 4. *To find the tangent of the curve* $2(x^2 + y^2)^2 = 25(x - y)^3$ *at* $(3, 1)$

Differentiating,

$$4(x^2 + y^2)\frac{d}{dx}(x^2 + y^2) = 75(x - y)^2\frac{d}{dx}(x - y)$$

$$8(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) = 75(x - y)^2\left(1 - \frac{dy}{dx}\right)$$

This gives the gradient at $(x, y)$. Substituting $x = 3$, $y = 1$, the gradient $m$ of the tangent is found to be $3/19$. The equation of the tangent is

$$y - 1 = 3(x - 3)/19$$
$$3x - 19y + 10 = 0$$

**72. Subtangent and subnormal.**

We take $P$ a point on a curve, $NP$ its ordinate, $TP$ its tangent, and $PG$ its normal. Then $TN$ is called the *subtangent* and $NG$ the *subnormal*. In the diagram a standard portion of a curve is drawn, in which $x$, $f(x)$, $f'(x)$ are all positive.

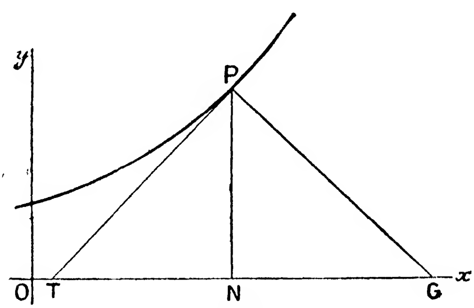



FIG. 35.

We denote the angle $NTP$ by $\psi$, so that $\tan\psi = f'(x)$, $x$ being the abscissa, $y = f(x)$. Then

$$TN = NP\cot\psi = y\cot\psi = f(x)/f'(x) = y\Big/\frac{dy}{dx}$$

$$NG = y\tan NPG = y\tan\psi = f(x)\,.\,f'(x) = y\frac{dy}{dx}$$

**73. Parametric representation of a curve.**

The equation of a curve in Cartesian coordinates implies a correspondence between two sets of numbers, one of which consists of values of the abscissae of points on the curve and the other of their ordinates. But there are other geometrical magnitudes associated

with each point; thus, every point has a radius vector ($r$) and an angular coordinate ($\theta$), also the length of the arc ($s$) measured from a fixed point of the curve up to the point considered, and the gradient ($\tan \psi$) of the curve at that point may be regarded as coordinates, that is, as variables determined by the position of $P$. Concentrating our attention on the radius vector, there is a functional relation between $x$ and $r$, say $x = F_1(r)$, and, similarly, $y = F_2(r)$. The curve is completely represented by either of these equations,

$$x = F_1(r) \qquad y = F_2(r)$$

It may also be represented by them both; for if we eliminate $r$ between the equations, we must get the relation $f(x, y) = 0$, which we call the equation of the curve. The representation in terms of $r$ of the two coordinates is not the only representation of the kind which is possible; by selecting $\theta$, $s$, $\psi$, or some other variable which has a value at each point, we obtain other pairs of equations by which the curve may be represented, in each case the third variable is called a *parameter*.

Perhaps the most general method of representation is obtained by thinking of a particle which describes the curve once, and once only; we could then tabulate three sets of numbers, (1) the time ($t$), (2) the abscissa ($x$), and (3) the ordinate ($y$) of the point occupied by the particle at time $t$. The pairing of these would give

$$x = f_1(t) \qquad y = f_2(t)$$

The auxiliary functions vary, of course, according to the law of motion of the particle; they are, however, connected, for on eliminating $t$, we must obtain a functional relation constituting the equation of the curve, and this is independent of the rate of its description.

### 74. Value of dy/dx, when x and y are functions of a parameter

Let $x = f_1(t)$, $y = f_2(t)$.

Then $\qquad x + \delta x = f_1(t + \delta t) \qquad y + \delta y = f_2(t + \delta t)$

$$\frac{\delta y}{\delta x} = \frac{f_2(t + \delta t) - f_2(t)}{f_1(t + \delta t) - f_1(t)} = \frac{f_2(t + \delta t) - f_2(t)}{\delta t} \bigg/ \frac{f_1(t + \delta t) - f_1(t)}{\delta t}$$

If $\delta t \to 0$, then $\delta x \to 0$ and $\delta y \to 0$, and we deduce

$$\frac{dy}{dx} = \frac{f_2'(t)}{f_1'(t)} = \frac{dy}{dt} \bigg/ \frac{dx}{dt}$$

As a first illustration we take $y^2 = 4ax$, and write

$$x = at^2 \qquad y = 2at$$

The curve is obtained by giving $t$ a range of values from $-\infty$ to 8.

Every quantity associated with the curve can be expressed in terms of $t$, thus

$$r = at\sqrt{(4 + t^2)} \qquad \tan\theta = 2/t$$

also

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = 2a/2at = 1/t$$

Here $t$ may be called the cogradient, its value being $\cot\psi$.

A second important use of a parameter is in connection with the circle in which we take

$$x = a\cos\theta \qquad y = a\sin\theta$$

in this case

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = a\cos\theta/(-a\sin\theta) = -\cot\theta$$

which is geometrically evident.

For a third illustration we have

$$ay^2 = x^2(a - x)$$

here we write $x = a(1 - t^2)$, $y = at(1 - t^2)$.

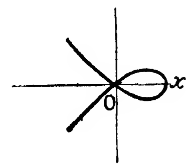


FIG. 36.

The range of $t$ is $[-\infty, \infty]$; the range $[-\infty, -1]$ corresponds to the part of the curve in the second quadrant, since $x$ is negative and $y$ is positive; $[-1, 0]$ gives the fourth quadrant part, while $[0, 1]$ gives the first quadrant, and when $t$ is in $[1, \infty]$ we get the part of the curve in the third quadrant.

**75. Equations of tangent and normal to the curve $x = f_1(t)$, $y = f_2(t)$.**

Now

$$\frac{dy}{dx} = \frac{f_2'(t)}{f_1'(t)}$$

It follows that the equations of the tangent and normal are respectively

$$f_1'(t)[y - f_2(t)] = f_2'(t)[x - f_1(t)]$$

and

$$f_2'(t)[y - f_2(t)] = -f_1'(t)[x - f_1(t)]$$

## EXERCISES VII

1. Write down the equations of the tangents of the following curves at the points mentioned

| | |
|---|---|
| i. $y^2 = 3x$ at $(3, 3)$ | ii. $xy = 6$ at $(-2, -3)$ |
| iii. $y = -\sqrt{(5 - 4x^2)}$ at $(1, -1)$ | iv. $y = \sqrt{(4x^2 + 5)}$ at $(1, 3)$ |
| v. $13y = x^3 - 3/x$ at $(3, 2)$ | vi. $y = (x + 1)/(2x - 1)$ at $(1, 2)$ |
| vii. $6y = x^3 - x$ at $(3, 4)$ | viii. $y = \sin x$ at $(\frac{1}{6}\pi, \frac{1}{2})$ |
| ix. $y = \tan^2 x$ at $(\frac{1}{4}\pi, 1)$ | x. $y = \sin x/x$ at $(\pi, 0)$ |

2. Prove that $2x + y = 2a$ is a tangent to $xy^2 = a^2(a - x)$, and show that it cuts the curve again at $(a, 0)$.

3. Show that the tangent of $27ay^2 = 4x^3$ at $(3a, -2a)$ is normal to $y^2 = 4a(x + 2a)$ at $(-a, 2a)$.

4. Find the equation of the tangent at $(a, b)$ to

$$(x/a)^n + (y/b)^n = 2$$

Determine also the points other than points on the axes at which the normals go through the origin, distinguishing the cases of $n$ even and odd.

5. Find the equations of the tangents and normals of

$$(x^2 + a^2)(y - a) = 2a^2x$$

at the points $(0, a)$ and $(a, 2a)$.

6. Prove that the tangent to $x^3 + 2xy^2 - y^3 = 17$ at $(2, 3)$ cuts $Oy$ at an angle of about $5° 42'$.

7. Show that $36x + 45y = 14a$ cuts $x^3 + y^3 = ax^2$ at right angles at $(\frac{1}{9}a, \frac{2}{9}a)$.

8. Find the abscissae of the points of contact of the tangents of

$$y = 8x^3 - 44x^2 + 78x - 45$$

which are parallel to $y + 2x = 0$.

9. Show that $x^2 + y^2 = 144$ cuts $y^2 = 10x$ at an angle equal to about $71° 1'$.

10. The tangent at $P$ to $y^m = kx^n$ meets $Ox$ at $T$. Prove that

$$m \tan xTP = n \tan xOP$$

11. The tangent at $P$ to $y^2 = kx^3$ meets the curve again at $Q$. Prove that

$$\tan xOP = 2 \tan QOx$$

12. The tangent at $P$ to $a^ny^m = b^mx^n$ meets the axes in $T$ and $t$. Prove that the ratio of $PT : Pt$ is constant.

13. Show that the length of the perpendicular from $O$ upon the tangent at $P$ $(x, y)$ is

$$\left(y - x\frac{dy}{dx}\right)\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-\frac{1}{2}}$$

Prove that in the rectangular hyperbola this perpendicular varies inversely as $OP$.

14. Give the Cartesian equation of the curve given by

$$x = a\cos^3 t \qquad y = a\sin^3 t$$

Show that the tangent at $t$ is

$$x \sin t + y \cos t = a \sin t \cos t$$

15. Prove that $x \cos t - y \sin t = a \cos 2t$ is a normal to the curve given by

$$x = a\cos^3 t \qquad y = a\sin^3 t$$

Show that the locus of the feet of the perpendiculars from $O$ upon these normals is $r = a \cos 2\theta$.

16. Prove that $2x - 3ty + at^3 = 0$ touches $y^3 = ax^2$.

17. Show that $x = a \cos 2t$, $y = a \sin t$ gives a portion of a parabola, and that $x + 4y \sin t = a(2 - \cos 2t)$ is a tangent to this given portion.

18. Find the equation of the tangent at $t$ to the curve given by

$$x = at/(1 + t^2) \qquad y = at^2/(1 + t^2)$$

What are the tangents at $t = 0$ and $t = \infty$ ?

19. A curve is given by

$$x = a(2 \cos t + \cos 2t) \qquad y = a(2 \sin t - \sin 2t)$$

Show that the equation of the normal at $t$ is

$$x \cos \tfrac{1}{2}t - y \sin \tfrac{1}{2}t = 3a \cos \tfrac{3}{2}t$$

# CHAPTER VIII

## SECOND DIFFERENTIAL COEFFICIENT

### 76. Definitions.

The differential coefficient of $f(x)$ has been written in two forms

$$f'(x) \qquad \frac{d}{dx}f(x)$$

When considered as a function of $x$, $f'(x)$ is called the derived function. It is our purpose in this chapter to study the differential coefficient of the derived function. The totality of the values of these differential coefficients is called the *second derived function of* $f(x)$. The differential coefficient of the derived function is termed the *second differential coefficient of* $f(x)$. The notations appropriate to these new conceptions are based upon the notations already used for the derived function and the differential coefficient. The second derived function may be written in either of the expressions

$$f''(x) \qquad \frac{d}{dx}f'(x)$$

There are advantages in the second expression, as by its form it separates into two stages the difficulties which beset the generation o $f''(x)$ from $f(x)$. For in any attempt to define $f''(x)$ from $f(x)$ we have to meet the difficulties which surround the definition of $f'(x)$ as well as those which occur in defining

$$\frac{d}{dx}f'(x)$$

By presupposing the existence of $f'(x)$ the difficulties are encountered singly. To define the second derived function, we must postulate (i) that $f'(x)$ exists in an open range, situated within the range of definition of $f(x)$, and (ii) that

$$\frac{f'(x+h) - f'(x)}{h}$$

approaches uniquely a finite limit as $h \to 0$; if these conditions are complied with, we may write this limiting value as

$$\frac{d}{dx}f'(x) \quad \text{or} \quad f''(x)$$

It should be noted that the second condition implies the continuity of $f'(x)$.

The corresponding notation for the second differential coefficient is built up on the analogy of the forms by which we represented the first differential coefficient, namely,

$$\frac{dy}{dx} = \frac{d}{dx}f(x) = f'(x)$$

Thus the second differential coefficient is

$$\frac{d}{dx}\cdot\frac{dy}{dx} = \frac{d}{dx}f'(x) = f''(x)$$

the first expression is written more succinctly as

$$\frac{d^2y}{dx^2}$$

an expression which is read as '*d* two *y* by *dx* squared.'

### 77. Symbol D for differentiation.

We may now introduce a further simplification in our symbols, by writing D for the operation of differentiation. Thus, in the case of a single differentiation, we write

$$\mathrm{D}f(x) = f'(x) \qquad \mathrm{D}y = \frac{dy}{dx} = \frac{d}{dx}f(x)$$

and in the case of a second differentiation

$$\mathrm{D}^2f(x) = \mathrm{D}\,.\,\mathrm{D}f(x) = \mathrm{D}f'(x) = f''(x)$$

$$\mathrm{D}^2y = \mathrm{D}\frac{dy}{dx} = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

Further applications of the use of D are given in the following chapter; but at the present stage it may be regarded as a short way of writing an operation. Sometimes a letter is suffixed to indicate the variable with respect to which the differentiation is effected; thus $\mathrm{D}_xy$ signifies the differential coefficient of $y$ with regard to $x$. The result of Art. 62 would be written

$$\mathrm{D}_xy\,.\,\mathrm{D}_yx = 1$$

and that of Art. 74 would be

$$\mathrm{D}_xy = \mathrm{D}_ty/\mathrm{D}_tx$$

It is permissible, with requisite safeguards, to extend the process of differentiation and to obtain differential coefficients of higher order; thus, we have

$$\mathrm{D}^3y = \mathrm{D}\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3}, \quad \mathrm{D}^4y = \frac{d^4y}{dx^4}, \ldots \mathrm{D}^ny = \frac{d^ny}{dx^n}$$

It is clear that two of the laws of positive integral indices hold for powers of D. Thus

$$D^m D^n y = D^{m+n} y$$

and

$$(D^m)^n y = D^{mn} y$$

**78. Geometrical meaning of the sign of the second differential coefficient.**

The argument of Art. 42 may be repeated by the substitution of $f'(x)$ for $f(x)$. Thus, if $f'(x)$ increases with $x$, $Df'(x)$ is positive; again, if $f'(x)$ decreases with $x$, $Df'(x)$ is negative; also $f''(x) = 0$ gives the values of $x$ for which $f'(x)$ is stationary.

We now consider the geometrical properties of the graph of $f(x)$, which are indicated by the sign of $f''(x)$. These could be deduced from diagrams such as Fig. 13, p. 52, where typical graphs of $f(x)$ and $f'(x)$ illustrate this problem. For the interest of the question we follow another course, and consider the rotation of the tangent as $P$, its point of contact, describes the curve.

Let us draw lines $Op$, $Oq$ parallel to the tangents at $P$, $Q$ in Figs. 37*a*, 37*b*. Although in both curves there is an up grade,

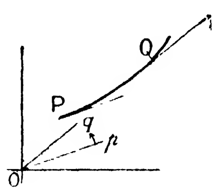



FIG. 37*a*.

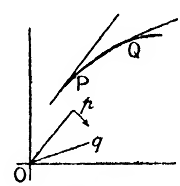



FIG. 37*b*.

the tangent is in the two diagrams rotating in contrary directions. In the left-hand diagram the tangent, and therefore $Op$, is rotating counter-clockwise, while in the second the direction is clockwise. The obvious geometrical distinction between the curves in the two diagrams is that the left-hand curve is concave to an observer situated at a great distance above $Ox$, and that the right-hand curve is convex. Thus, concavity at $P$ in Fig. 37*a* corresponds to the fact that at $P$ the tangent is rotating counter-clockwise, as the abscissa of $P$ increases.

Now, in a curve such as the graph of the quadratic function, the tangent is always rotating in the same direction: the curve, therefore, is concave or convex throughout; but other curves which are the graphs of single-valued functions may change from concave to convex or from convex to concave, as $x$ increases; an example of such a curve is the well-known sine-curve. Let us see what happens at the junction of convex and concave portions.

Let us take a curve $APB$, in which the arc $AP$ is concave upwards

and $PB$ is convex, and study the rotation of the tangent as its point of contact moves from $A$ to $B$. The tangent revolves ↺ from $Oa$ to $Op$ and then ↻ from $Op$ to $Ob$. The gradient of the tangent at $P$ is a maximum. The tangent at $P$ differs from an ordinary tangent, inasmuch as in the neighbourhood of $P$ the arc $AP$ lies above the tangent and the arc $PB$ lies below it; the curve indeed at $P$ crosses its tangent—such a point is called *a point of inflexion*. There is also a point of inflexion at which the gradient is a minimum; such a point separates a convex portion on the left from a concave portion on the right. The student should draw a figure to illustrate this case.

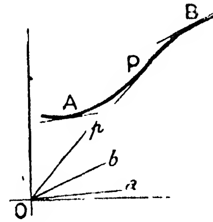



FIG. 38.

The conic sections have no points of inflexion, but such points are familiar to the skater as points on the ice-curves described by his skate at which he changes his edge; another example is afforded by the familiar figure 8.

**79. Relation between the graphs of f(x), f′(x) and f″(x).**

The relation between the graphs of $f(x)$ and $f'(x)$ has been discussed in Art. 46. It remains to discuss the relation between those of $f'(x)$ and $f''(x)$ and to see what properties of the graph of $f(x)$ are revealed by properties of the graph of $f''(x)$.

Now, when $f''(x)$ is positive, $f'(x)$ is increasing and the graph of $f(x)$ is concave in the sense explained above, while if $f''(x)$ is negative,

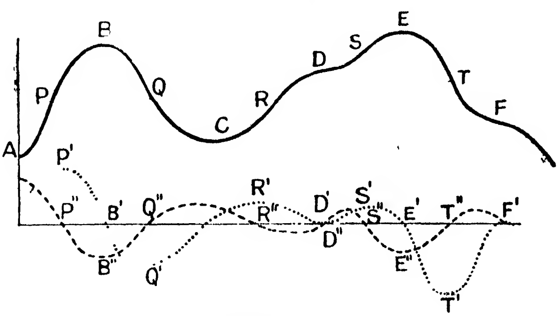



FIG. 39.

the graph of $f(x)$ is convex. Also at a node of the graph of $f''(x)$ we have a point of inflexion of the graph of $f(x)$. Again, at an ascending node of the graph of $f''(x)$, the graph of $f(x)$ changes from convex to concave, and at a descending node, the graph of $f(x)$ changes from concave to convex.

Another combination is of interest; the graph of $f'(x)$ may touch $Ox$ but not cross it. In this case the graph of $f''(x)$ passes through

the point of contact of the graph of $f'(x)$ and the graph of $f(x)$ has a point of inflexion at which the tangent is parallel to $Ox$.

**80. Criteria for maxima and minima.**

It is easy now to give a second set of criteria for stationary values which are maxima or minima. We see that

(i) if $f'(a) = 0$ and $f''(a)$ is positive, then $f(a)$ is a minimum ;

(ii) if $f'(a) = 0$ and $f''(a)$ is negative, then $f(a)$ is a maximum.

But if $f''(a) = 0$ we cannot discriminate the stationary values without reference to higher differential coefficients.

**81. Illustrations of f″(x).**

First, we take the quadratic function

$$\begin{aligned} f(x) &= ax^2 + 2bx + c \\ f'(x) &= 2(ax + b) \\ f''(x) &= 2a \end{aligned}$$

The graph has no point of inflexion, it is concave to an observer above $Ox$, if $a$ is positive ; convex, if $a$ is negative.

Secondly, we take the cubic function

$$\begin{aligned} f(x) &= ax^3 + 3bx^2 + 3cx + d \\ f'(x) &= 3(ax^2 + 2bx + c) \\ f''(x) &= 6(ax + b) \end{aligned}$$

The graph of the cubic function has a single point of inflexion which lies upon $x = -b/a$: this line divides the graph into two parts, one of which is concave and the other is convex. Further, if $b^2 = ac$, the inflexional tangent is parallel to $Ox$, because in this case $f'(-b/a) = 0$.

Just as the quadratic function was reduced to the form $ax'^2 + D/a$ by writing $x' = x + b/a$, that is, by taking the origin on the axis of the parabolic graph ; so, by choosing $x' = x + b/a$, the cubic function may be reduced to a form in which the second term is absent. In the reduced form of the cubic the new axis of $y$ passes through the point of inflexion.

**82. A parabolic approximation to y = f(x).**

In Art. 68 we determined a linear function

$$ax + b$$

so that its graph passed through a point of

$$y = f(x)$$

and had the same differential coefficient as that of $f(x)$ at the given value. So we can now determine a quadratic function

$$ax^2 + 2bx + c$$

such that its graph passes through the point $(x_1, f(x_1))$ and has the same first and second differential coefficients there as the given curve. It follows that

$$f(x_1) = ax_1^2 + 2bx_1 + c$$
$$f'(x_1) = 2(ax_1 + b)$$
$$f''(x_1) = 2a$$

and these three equations provide means of determining $a$, $b$, $c$ in terms of $x_1$, $f(x_1)$, $f'(x_1)$ and $f''(x_1)$. Geometrically, this means the determination at $x = x_1$ of the parabola of closest contact which has its axis parallel to $Oy$.

**83. The circle of curvature.**

It is more convenient to compare the curve $y = f(x)$ at each point with a circle than with a parabola. The comparison still involves the determination of three constants, namely, the radius and the coordinates of the centre; these are settled in the same way as in Art. 82 by making the circle pass through the point considered and have values of $\mathrm{D}y$, $\mathrm{D}^2y$ equal to those obtained from the equation of the curve at the point.

Let the equation of the circle be

$$(x - h)^2 + (y - k)^2 = c^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $(h, k)$ is the centre and $c$ is the radius; then by differentiation we have

$$(x - h) + (y - k)\frac{dy}{dx} = 0 \quad . \quad . \quad . \quad . \quad (2)$$

and by a second differentiation

$$1 + \frac{d}{dx}(y - k) \,.\, \frac{dy}{dx} + (y - k)\frac{d^2y}{dx^2} = 0$$

that is,

$$1 + \left(\frac{dy}{dx}\right)^2 + (y - k)\frac{d^2y}{dx^2} = 0 \quad . \quad . \quad . \quad . \quad (3)$$

Now, from equations (1), (2) and (3) we find $h$, $k$, $c$ in terms of $x$, $y$, $\mathrm{D}y$, $\mathrm{D}^2y$; thus, for $h$, $k$ we have

$$x - h = \left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}\Big/\frac{d^2y}{dx^2} \qquad y - k = -\left[1 + \left(\frac{dy}{dx}\right)^2\right]\Big/\frac{d^2y}{dx^2}$$

and substituting in (1),

$$c^2\left[\frac{d^2y}{dx^2}\right]^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right]$$

hence

$$c = \pm\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}\Big/\frac{d^2y}{dx^2}$$

To make $c$ positive, we select the upper sign if $\mathrm{D}^2y$ is positive, and the lower sign if $\mathrm{D}^2y$ is negative. The circle whose centre $(h, k)$ and radius $c$ are thus determined is called the *circle of curvature* at $(x, y)$.

## 84. Curvature.

A brief statement, which will be supplemented in Section II, is made of the meaning of curvature of a curve at $(x, y)$.

In a circle the curvature (or bending) at every point is the same; this is an inference from symmetry. Again, in a large circle the curvature at each point is small; in a small circle it is large. We say that *the reciprocal of the radius of a circle is a measure of its curvature.* Also the reciprocal of the radius of the circle of curvature at a point $(x, y)$ is a measure of the curvature of the curve at this point.

## 85. Formulae for curvature.

Although the radius of a circle is a positive quantity, it is convenient to give the curvature the sign of $D^2y$ and take for its value

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

With this convention we see that positive curvature implies that the curve is concave upwards, and negative curvature implies that it is convex. We may notice also that at a point of inflexion the curvature is zero.

The formula for curvature assumes a very simple form when the tangent at $P$ is parallel to $Ox$; in this case the curvature is

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

Further, in the case in which the gradient is small throughout, as, for instance, in a beam suspended by its middle point, the curvature may be taken as equal to $$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

since in the denominator of the expression for curvature the gradient is small and only its square appears.

## 86. Illustrative examples.

Ex. 1. *To find the radius of curvature at any point of the parabola* $y = x^2/4a$.

We have $Dy = x/2a$, $D^2y = 1/2a$ and

$$1 + (Dy)^2 = 1 + x^2/4a^2 = 1 + y/a$$

hence the radius of curvature $= 2(a + y)^{\frac{3}{2}}/a^{\frac{1}{2}}$.

Since in the parabola $a + y = SP$, the square of the radius of curvature varies as the cube of the focal distance.

The diagram, Fig. 40, represents circles of curvature, the abscissae of whose points of contact are $0, a, 2a, 3a, \ldots$, the centres of the circles of curvature, or centres of curvature, being at $C_0, C_1, C_2, \ldots$; the parabola, though not drawn, is suggested by these circles of curvature.

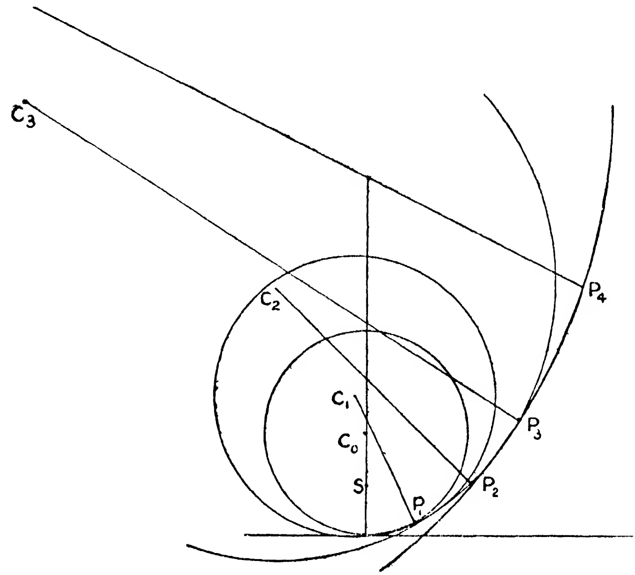



FIG. 40.

Ex. 2. *To find the radius of curvature at any point of the ellipse*

$$x^2/a^2 + y^2/b^2 = 1$$

By differentiating $\qquad b^2x^2 + a^2y^2 = a^2b^2$

we have

$$b^2x + a^2y\frac{dy}{dx} = 0$$

also

$$b^2 + a^2\left(\frac{dy}{dx}\right)^2 + a^2y\frac{d^2y}{dx^2} = 0$$

whence

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y} \qquad -\frac{d^2y}{dx^2} = \frac{b^4}{a^2y^3}$$

The radius of curvature is equal to

$$(a^4y^2 + b^4x^2)^{\frac{3}{2}}/a^4b^4 = CD^3/ab$$

where $CD$ is the semi-diameter parallel to the tangent at $(x, y)$.

Ex. 3. *To find the radius of curvature at the point* (1, 2) *of the curve whose equation is*

$$y^2 = x^3 + 3x^2$$

By differentiation we have

$$2y\frac{dy}{dx} = 3x^2 + 6x$$

and
$$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 6x + 6$$

Writing $x = 1$, $y = 2$, we find

$$\frac{dy}{dx} = \frac{9}{4} \qquad \frac{d^2y}{dx^2} = \frac{15}{32}$$

whence
$$c = 97^{\frac{3}{2}}/30 = 31{\cdot}85$$

**87. The orders of magnitudes.**

There is no absolute standard by which a particular quantity can be said to be small. Thus, in measuring the radius of the earth a mile is small, while in measuring a man we take account of the quarter of an inch. In the infinitesimal calculus it is with the ratio of $\delta y : \delta x$, where $x$ and $y$ are functionally related, that we are so often concerned; these small quantities of the calculus have, however, a property which renders them particularly difficult to grasp; it is their liability to variation. For when we are discussing the ratio of $\delta y : \delta x$, we must think of the ratio of the terms of a sequence of $\delta y$'s to the corresponding terms of a sequence of $\delta x$'s. Let us suppose that the limiting value of $\delta y : \delta x$ is $m$, and let us consider the sequence $\delta_1 x$, $\delta_2 x$, $\delta_3 x$, ..., and the consequent sequence of the increments of $y$, $\delta_1 y$, $\delta_2 y$, $\delta_3 y$, ...; then we can write

$$\frac{\delta_1 y}{\delta_1 x} = m + \varepsilon_1 \qquad \frac{\delta_2 y}{\delta_2 x} = m + \varepsilon_2 \qquad \frac{\delta_3 y}{\delta_3 x} = m + \varepsilon_3, \ldots$$

where the $\varepsilon$'s are small compared to $m$, and by the nature of $m$, the $\varepsilon$'s form a sequence whose limit is zero. In this case we say that the small increments $\delta x$, $\delta y$ are of the same order.

Two cases have to be further considered, namely, when $m = 0$, and when $1/m = 0$. Under either of these conditions $\delta x$ and $\delta y$ are of different orders.

If $m = 0$, then, we may find that

$$\frac{\delta y}{(\delta x)^2} = n + \varepsilon$$

where $n = \lim \delta y/(\delta x)^2$ and $\lim \varepsilon = 0$, $\delta x \to 0$; in this case, the order of $\delta y$ is double that of $\delta x$. Generally, if

$$\frac{(\delta y)^p}{(\delta x)^q} = k + \varepsilon$$

where $k$ is finite and not zero and $\varepsilon \to 0$, when $\delta x \to 0$, we say that $\delta x$ is of the $p$th order of small quantities and $\delta y$ of the $q$th order. By this we imply that if the typical small quantity is $10^{-r}$, $\delta y$ is of

the order $10^{-rq}$, and $\delta x$ of the order $10^{-rp}$, whence it follows that $(\delta y)^p$ and $(\delta x)^q$ are of the same order, namely, $10^{-rpq}$.

As a geometrical illustration let us take a curve which passes through $O$ and represents a functional relation between $x$ and $y$. Then $(\delta x, \delta y)$ is a point near $O$. If

$$\frac{\delta y}{\delta x} = m + \varepsilon$$

the tangent at $O$ is inclined to $Ox$ at an angle $\tan^{-1} m$. But if $m = 0$, $Ox$ is the tangent at $O$, while if $1/m = 0$, $Oy$ is the tangent at $O$; in these cases $\delta x$, $\delta y$ are of different orders, and their respective orders depend upon the equation of the curve.

If the equation of the curve is (1) $y^2 = ax$, then $(\delta y)^2 = a\,\delta x$, and $\delta x$ being of the second order, $\delta y$ is of the first order (Fig. 41$a$), while if (2) it is $x^2 = ay$, $\delta x$ is of the first order and $\delta y$ of the second (Fig. 41$b$);

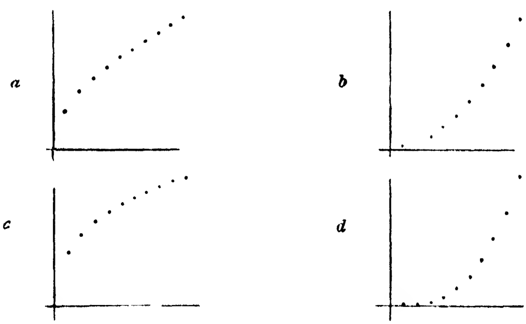



FIG 41.

again, if (3) it is $y^3 = a^2x$, then, $\delta y$ being of the first order, $\delta x$ is of the third order (Fig. 41$c$), and if (4) $x^3 = a^2y$, $\delta x$ being of the first order, $\delta y$ is of the third order. The diagrams give points whose abscissae are 0·1, 0·2, ... , $a$ being taken unity.

Further information on the geometrical form of a curve at a point is given in Chapter XVII.

We may also illustrate the important fact that there are different orders of large magnitudes by taking the equation of the hyperbola referred to a vertex as origin and by supposing that the lengths of the axes of the hyperbola become infinitely large, while its eccentricity approaches unity as a limit, deducing from it the equation of a parabola.

The equation of the hyperbola referred to an end of the transverse axis as origin is

$$(x + a)^2/a^2 - y^2/b^2 = 1$$

that is,

$$y^2 = 2xb^2/a + x^2b^2/a^2$$

and

$$e^2 = (a^2 + b^2)/a^2 = 1 + b^2/a^2$$

Now, if $a \to \infty$, $b \to \infty$, and $e \to 1$, we must have $b^2/a^2 \to 0$, so that $b/a \to 0$; this requires that $a$, $b$ are of different orders. If we take

that $b^2/a \to p$, a finite magnitude, we have as the equation of the curve approached

$$y^2 = 2xp$$

The conditions imposed mean that while $b$ is of the first order of large quantities, $a$ is of the second order.

**88. Velocity and acceleration.**

If $x$ is the coordinate which determines the position upon $Ox$ at time $t$ of a particle describing this axis, and if $x + \delta x$ gives its position at time $t + \delta t$, then $\delta x/\delta t$ is its average velocity in the interval $\delta t$. Again, the velocity at time $t$ is $\lim \delta x/\delta t$, $\delta t \to 0$. We write this limit

$$\frac{dx}{dt} = v$$

The notation of the differential coefficient is naturally adapted to the measurement of velocity.

Again, if $v + \delta v$ is the velocity at time $t + \delta t$, the average acceleration in the interval $\delta t$ is $\delta v/\delta t$, and we have

$$\frac{dv}{dt} = f$$

where $f$ is the acceleration at time $t$.

Also we have, in the notation of Art. 76,

$$f = \frac{dv}{dt} = \frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2}$$

Valuable illustrations of the calculus are derived by studying the graphs of $x$ and $v$ considered as functions of $t$. The first is called the displacement-time graph, and the second is the velocity-time graph. They are connected with each other by the relations which unite the characteristics of a function and its derived function. When the velocity ($v$) vanishes, the particle is at a stationary point; if $v$ not only vanishes but changes sign, the particle is at a maximum or a minimum distance from $O$. Further, if the graph of $f$ is drawn, we see that, at the stationary points of the $v$-graph, the acceleration of the particle vanishes, and that at the corresponding point of the $x$-graph there is a point of inflexion.

Turning to the diagram on p. 95, the reader will see that if times indicated by the abscissae of $B, C, \ldots$ be denoted by the corresponding small letters, the velocity vanishes at times $b, c, d, e, f, \ldots$, the acceleration vanishes at times $p, q, r, d, s, t, \ldots$. The point $D$, which is a stationary point and a point of inflexion, corresponds to a *dead* point of the motion, since there is no velocity and no acceleration.

**89. Leibniz's theorem.**

If $u, v$ are functions of $x$, we have, from Art. 35,

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

Or, as it may be written,

$$D(uv) = uv' + u'v$$

where dashes denote differentiation with respect to the variable $x$.

This theorem can be generalised, and we can find $D^n(uv)$. First we find $D^2(uv)$, thus

$$\begin{aligned} D^2(uv) &= D(uv') + D(u'v) \\ &= uv'' + u'v' + u'v' + u''v \\ &= uv'' + 2u'v' + u''v \end{aligned}$$

Let us assume that

$$\begin{aligned} D^{n-1}(uv) = uv^{(n-1)} + (n-1)_1 u'v^{(n-2)} + (n-1)_2 u''v^{(n-3)} + \dots \\ + (n-1)_1 u^{(n-2)}v' + u^{(n-1)}v \end{aligned}$$

where $n_r$ is the number of combinations of $n$ different things taken $r$ together.

Now, on differentiating each side of this identity, we have *

$$\begin{aligned} D^n(uv) &= uv^{(n)} + u'v^{(n-1)} + (n-1)_1(u'v^{(n-1)} + u''v^{(n-2)}) \\ &\quad + (n-1)_2(u''v^{(n-2)} + u'''v^{(n-3)}) + \dots + u^{(n-1)}v' + u^{(n)}v \\ &= uv^{(n)} + \left[1 + (n-1)_1\right] u'v^{(n-1)} \\ &\quad + \left[(n-1)_1 + (n-1)_2\right] u''v^{(n-2)} + \dots + u^{(n)}v \\ &= uv^{(n)} + n_1 u'v^{(n-1)} + n_2 u''v^{(n-2)} + \dots + u^{(n)}v \end{aligned}$$

The result thus established by induction is named after Leibniz, one of the great thinkers who laid the foundations of the infinitesimal calculus.

## EXERCISES VIII (A)

### Second differential coefficients, points of inflexion

1. If $y = (x-1)^{-1}$ $\qquad \dfrac{d^2y}{dx^2} = 2(x-1)^{-3}$

2. If $y = \dfrac{ax^2 + bx + c}{x - 1}$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{2(a+b+c)}{(x-1)^3}$

3. If $y = \dfrac{3x}{(x+1)(x-2)}$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{2}{(x+1)^3} + \dfrac{4}{(x-2)^3}$

4. If $y = \dfrac{ax^2 + bx + c}{(x-p)(x-q)}$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{2(ap^2 + bp + c)}{(p-q)(x-p)^3} - \dfrac{2(aq^2 + bq + c)}{(p-q)(x-q)^3}$

* The theorem used in reducing the coefficients is

$$n_r = (n-1)_r + (n-1)_{r-1}$$

a result which is directly proved by considering the combinations of the group of $n$ different things as compared with those which are made first by excluding a particular thing and then by including it.

5. If $y = \sqrt{(1 + x^2)}$ $\qquad \dfrac{d^2y}{dx^2} = (1 + x^2)^{-\frac{3}{2}}$

6. If $y = x\sin x + \cos x$ $\qquad \dfrac{d^2y}{dx^2} = \cos x - x\sin x$

7. If $y = \dfrac{1 + \sin x}{\cos x}$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1 - \sin x)^2}$

8. If $y = A\sin x + B\cos x$ $\qquad \dfrac{d^2y}{dx^2} = -y$

9. If $y^2 = \sec 2x$ $\qquad \dfrac{d^2y}{dx^2} = 3y^5 - y$

10. If $y^2 = ax^2 + b$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{ab}{y^3}$

11. If $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ $\qquad \dfrac{d^2y}{dx^2} = \dfrac{1}{3}a^{\frac{2}{3}}x^{-\frac{4}{3}}y^{-\frac{1}{3}}$

12. Show that $\qquad y = 27x^3 - 54x^2 + 36x - 8$
has a point of inflexion on $Ox$.

13. Show that $\qquad y = x^4 - 6x^2 + 5x$
has points of inflexion at (1, 0) and (– 1, – 10).

14. Determine $f(x)$, a cubic function of $x$, such that $y = f(x)$ has a point of inflexion at (– 1, 16) and touches $Ox$ at (1, 0).

*Ans.* $x^3 + 3x^2 - 9x + 5$.

15. Determine $f(x)$, a quartic function of $x$, such that $y = f(x)$ passes through $O$ and has points of inflexion at (1, 7) and (– 1, – 17).

*Ans.* $x^4 - 6x^2 + 12x$.

16. Show that the points of inflexion of

$$y = 4\sin x - \sin 2x$$

are at points whose abscissae are $n\pi$, $(2n \pm \frac{1}{3})\pi$, where $n$ is integral.

17. Show that the origin, $(\sqrt{3}a, \frac{1}{4}a\sqrt{3})$ and $(-\sqrt{3}a, -\frac{1}{4}a\sqrt{3})$ are the points of inflexion of

$$y(a^2 + x^2) = a^2x$$

18. Prove that the abscissae of the points of inflexion of $y^2(1 + x^4) = x^2$ are $x = 0, \pm \sqrt[4]{5}$.

19. Show that $y = xy + y^2 + x^3$ has a point of inflexion at $O$.

20. Show that $y = (x^3 - a^3)/ax$ has a point of inflexion at the point at which it cuts $Ox$.

21. If $x = \cos t + t\sin t$, $y = \sin t - t\cos t$, prove that

$$\frac{dy}{dx} = \tan t \qquad \frac{d^2y}{dx^2} = \frac{\sec^3 t}{t}$$

22. If $x = t - t^{-1}$, $y = t^m - t^{-m}$, prove that

$$(x^2 + 4)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$$

23. If $x = \sin y$, show that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

24. If $x = \sin\sqrt{y}$, show that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$$

25. By applying Leibniz's theorem to the case in which $u = x^n$, $v = x^n$, deduce that the sum of the squares of the coefficients in the expansion of $(1 + x)^n$ is equal to $(2n)!/(n!)^2$.

26. Find the $n$th differential coefficient of $x^n \log x$.

$$Ans.\ n!\left[\log x + n - \frac{1}{2}\,\frac{n(n-1)}{2!} + \frac{1}{3}\,\frac{n(n-1)(n-2)}{3!} - \dots\right]$$

27. Show that

i. $D^n xf(x) = xf^{(n)}(x) + nf^{(n-1)}(x)$

ii. $D^n x^2 f(x) = x^2 f^{(n)}(x) + 2nxf^{(n-1)}(x) + n(n-1)f^{(n-2)}(x)$

28. Prove that

i. $D^{4n} x \cos x = x \cos x + 4n \sin x$

ii. $D^{4n+1} x^2 \sin x = x^2 \cos x + (8n + 2)\, x \sin x - (16n^2 + 4n) \cos x$

29. Prove that if $x$ and $y$ are connected either by the relation given in Question 23 or by that in 24,

$$(1 - x^2)y^{(n+2)} - (2n + 1)\, xy^{(n+1)} - n^2 y^{(n)} = 0$$

## EXERCISES VIII (B)

### Radius of curvature

1. Construct the circles of curvature of $y^2 = x$ at the points (0, 0), (1, 1), (4, 2), ... by calculating the radii of curvature at these points and determining the corresponding centres of curvature.

2. Compare the radii of curvature of the curves

$$y^2 = x^3 \qquad y = x^3$$

at the three points whose abscissae are 1, $\frac{1}{2}$, $\frac{1}{10}$. Show that at the origin the radius of curvature of the first curve is zero and of the second is $\infty$.

3. Draw the circle of curvature of the curve

$$y = x + 1/x$$

at the point (1, 2), and show that $\rho = \frac{1}{2}$. Prove that the parabola of closest contact at this point with axis parallel to $Oy$ is $y = x^2 - 2x + 3$. Draw a diagram showing this parabola in its relation to the curve.

4. Sketch the curve $\qquad y = 2x^2 - x^3$

and find the circles of curvature at the points given by $x = -1, 0, \frac{4}{3}, 2$.

5. Draw the parabolas

$$y = -x^2 + x + 1 \qquad x = -y^2 + y + 1$$

and show that they have the same circle of curvature at (1, 1), namely $x^2 + y^2 = 2$.

6. Prove that the parabola

$$2y = (y_1 - 2y_2 + y_3)(x/h)^2 - (y_1 - y_3)(x/h) + 2y_2$$

passes through $(-h, y_1)(0, y_2)(h, y_3)$. Deduce that at $(0, y_2)$

$$Dy = -\tfrac{1}{2}(y_1 - y_3)/h \qquad D^2y = (y_1 - 2y_2 + y_3)/h^2$$

7. The coordinates of three points $P$, $Q$, $R$ on a curve are

$$(0{\cdot}2125,\ 0{\cdot}8734) \qquad (0{\cdot}2250,\ 0{\cdot}9321) \qquad (0{\cdot}2375,\ 0{\cdot}9914)$$

Find approximate values of $Dy$, $D^2y$ at $Q$, and deduce that the radius of the circle $PQR$ is 29·2 nearly.

8. Given three points $P$, $Q$, $R$ on a curve

$$(a, y_1) \qquad (a + h, y_2) \qquad (a + h + k, y_3)$$

show that an approximate value of the second differential coefficient at $P$ is

$$2\frac{ky_1 - (h + k)y_2 + hy_3}{hk(h + k)}$$

Given $h = 1$, $k = 2$, $y_1 = 0{\cdot}312$, $y_2 = 0{\cdot}451$, $y_3 = 0{\cdot}675$, show that the radius of the circle $PQR$ is about 57.

9. Show that the radius of curvature of

$$y = a \sin (x/b)$$

at $x = \frac{1}{2}\pi b$ is equal to $b^2/a$.

10. Show that the radius of curvature of $y = 4 \sin x - \sin 2x$ at $x = \frac{1}{2}\pi$ is $5\sqrt{5}/4$.

11. If the radius of curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $(a/\sqrt{2}, b/\sqrt{2})$ is $a$, prove that the eccentricity of the ellipse is $\sqrt{(3 - \sqrt{5})}$.

12. Prove that the radius of curvature at $(a, b)$ of $(x/a)^n + (y/b)^n = 2$ is equal to

$$\frac{(a^2 + b^2)^{\frac{3}{2}}}{2(n - 1)ab}$$

13. Show that $x^2 + y^2 = 2$, $x^4 + y^4 = 2$ touch at (1, 1) and that their curvatures at this point are in the ratio of 1 : 3.

14. In the curve $y^2 = ax^2 + b$ show that

$$\rho = (y^2 + a^2x^2)^{\frac{3}{2}}/ab$$

15. In the curve $y^2(2a - x) = x^3$ show that

$$\rho = \frac{a\sqrt{x}(8a - 3x)^{\frac{3}{2}}}{3(2a - x)^2}$$

and that the centre of curvature is at

$$\left(\frac{ax(5x - 12a)}{3(2a - x)^2}, \frac{8}{3}\frac{ay}{x}\right)$$

16. In the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ prove that

$$\rho = 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}$$

17. Given a curve in which $x = \cos t + t \sin t$, $y = \sin t - t \cos t$, show that $\rho = t$.

18. Given that $x = \dfrac{t}{1 + t^2}$ $\quad y = \dfrac{t^2}{1 + t^2}$

show that $\dfrac{dy}{dx} = \dfrac{2t}{1 - t^2}$ $\quad \dfrac{d^2y}{dx^2} = \dfrac{2(1 + t^2)^3}{(1 - t^2)^3}$

Verify from the value of $\rho$ that the curve is a circle.

## EXERCISES VIII (c)

### Motion in a straight line

1. Sketch the displacement-time and velocity-time graphs of the following motions in one dimension, mentioning discontinuities in the second set of graphs,

i. A ball falling to the ground.

ii. A ball moving up and down a billiard table ($e = \frac{1}{2}$).

iii. A pendulum executing harmonic oscillations.

iv. A mass dropped into a bucket of water, considering the cases in which the mass is lighter or heavier than water.

v. A train going from one terminus to another and back, with a stop halfway.

2. Find the velocity at time $t = 0$ and the acceleration, given

$$x = at^2 + 2bt + c$$

3. Find the maximum velocity and the positions of instantaneous rest in the motion $\quad x = a(\sin nt + \cos nt)$

4. If the relation connecting displacement ($x$) and time ($t$) is

$$2\mu t = ax^2 + 2bx + c$$

show that the velocity varies inversely as the distance from the point $x = -b/a$ and that the acceleration varies as the cube of the distance from this point.

5. Given $v^2 = n^2(a^2 - x^2)$, show that the acceleration is directed towards $O$ and varies as the distance from $O$.

6. If the relation between $x$ and $t$ is

$$x = \frac{2t}{a^2 + t^2}$$

show that the velocity vanishes when $t = a$ and the acceleration when $t = a\sqrt{3}$.

# CHAPTER IX

## INVERSE DIFFERENTIATION

**90. General remarks.**

In an earlier chapter the relations between functions and their inverses were discussed. The simple functions, such as $x + 2$, $ax$, $x^2$, depend upon a single mathematical operation; the functions $x - 2$, $a/x$, $\sqrt{x}$ also imply a single operation, which may be termed the inverse of the former. Thus, addition and subtraction are inverse; so also are multiplication and division, and again, involution and evolution. The characteristic of inverse operations is that if they are successively applied to $x$, they reproduce $x$. Thus, if 2 is added to $x$, and then subtracted from the sum, the result is $x$; also, if $x$ is multiplied by $a$ and the product is divided by $a$, we get $x$; further, the square of the square root of $x$ gives $x$. In single-valued operations it is indifferent which of the operations is applied first; thus, not only does $(x - 2) + 2 = x$, but also $(x + 2) - 2 = x$.

The operations addition, multiplication and involution are called direct, while subtraction, division and evolution are inverse. The importance of these inverse operations may be realised by reflecting that negative numbers are due to the need of subtracting the larger of two positive numbers from the smaller, and that fractions are the outcome of our need to extend the realm of division.

As inverse operations are not only more difficult, but more tentative, we may expect, in turning to the process which is the inverse of differentiation, to find new difficulties and also to require considerable additions to be made to the list of functions which have been studied hitherto.

**91. Definition of inverse differentiation.**

The problem of inverse differentiation is expressed thus: Given a function of $x$, say $f(x)$, it is required to find a function whose differential coefficient with regard to $x$ is $f(x)$; that is, given $f(x)$, it is required to determine $y$, a function of $x$, to satisfy the equation

$$\frac{dy}{xd} = f(x)$$

dx

As a first illustration of the indefinite character of the answer we take $\varphi(x) = 2x$, and ask for functions which when differentiated give $2x$. The answers are numerous, and may include

$$x^2 \qquad x^2 + 1 \qquad x^2 - 2, \ldots$$

indeed, any function which differs from $x^2$ by a constant. The complete answer is represented by

$$y = x^2 + C$$

where C may be any constant or fixed number.

**92. Geometrical meaning of inverse differentiation.**

We have shown in Chap. IV the importance of studying the graph of the derived function $f'(x)$ side by side with that of $f(x)$. When we are given $f(x)$ there is a single definite derived function, but when we start with $f'(x)$ as given, there are many curves whose gradient at a point whose abscissa is $x$ is represented by $f'(x)$. For if one such curve is drawn, we can get any number of other curves by shifting this curve so that each point moves parallel to $Oy$ through the same distance.

Taking the illustration of Art. 91,

$$\frac{d}{dx}f(x) = 2x$$

we have $f(x) = x^2$ as one answer. The gradient at $P$ of the parabola $y = x^2$ is equal to the ordinate $NQ$, but obviously if the parabola is shifted up or down, parallel to $Oy$, the tangents at $P$ and $P'$ are parallel, and their gradients are both equal to $NQ$, that is

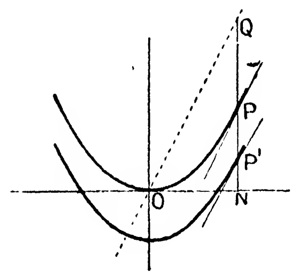



FIG. 42.

$$f(x) = x^2 + C$$

is also an answer to the question proposed, where C is the distance parallel to $Oy$ through which the parabola is shifted.

**93. Notation of an inverse function.**

The result of certain mathematical operations performed upon $x$ is denoted by $f(x)$, and there is an operation which, performed upon $f(x)$, reproduces $x$. This operation is denoted by

$$f^{-1}(x)$$

Then, on substituting $f(x)$ for $x$, we have

$$f^{-1}[f(x)] = x$$

which is in accord with the analogy of an index law.

The reader may compare the notation

$$\sin \sin^{-1} x = x \qquad \sin^{-1} \sin x = x$$

and should note that in the second equality a selection has to be made of the appropriate value of the incompletely defined function $\sin^{-1}$.

Again, if $f(x) = x^2$, $f^{-1}(x) = x^{\frac{1}{2}}$, we have

$$ff^{-1}(x) = f(x^{\frac{1}{2}}) = (x^{\frac{1}{2}})^2 = x$$

$$f^{-1}f(x) = f^{-1}(x^2) = (x^2)^{\frac{1}{2}} = x$$

selecting properly the sign of the incompletely defined square-root function; thus $f^{-1}f(-1) = f^{-1}(1) = 1^{\frac{1}{2}} = -1.$

Generally we have that

$$f[f^{-1}(x)] = x = f^{-1}[f(x)]$$

**94. Notation of inverse differentiation.**

We have used the notation

$$Df(x) = f'(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It is a ready deduction from this to say that

$$f(x) = D^{-1}f'(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again we have from (1), by applying $D^{-1}$ to each side,

$$D^{-1}Df(x) = D^{-1}f'(x) = f(x) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and by applying D to each side of (2),

$$DD^{-1}f'(x) = Df(x) = f'(x) \quad . \quad . \quad . \quad . \quad (4)$$

where the results in (3) and (4) are consistent with our general inverse notation.

We use $D^{-1}$ as a symbol for inverse differentiation, just as we have used D for differentiation, but it does not completely define an operation. Thus we may write

$$D^{-1}2x = x^2$$

as a consequence of $\quad Dx^2 = 2x$

but we also have $\quad D^{-1}2x = x^2 + 1$

as a consequence of $\quad D(x^2 + 1) = 2x.$

This incompleteness of definition is allowed for by writing

$$D^{-1}2x = x^2 + C$$

We shall suppose that in effecting $D^{-1}$ the constant is chosen so that the result of performing first D and then $D^{-1}$ upon a function reproduces the function. Symbolically we shall take $D^{-1}D = 1$ as well as $DD^{-1} = 1$.

The operation of inverse differentiation is usually termed ***integration***, and the result of integrating a function is spoken of as its ***integral***; sometimes the function to be integrated is called the ***integrand***.

**95. Integral of $x^n$, provided $n \neq -1$.**

Now $$D\frac{x^{n+1}}{n+1} = x^n$$

It follows that $$D^{-1}x^n = \frac{x^{n+1}}{n+1}$$

A rule may be expressed thus: *to integrate $x^n$, add unity to the index and divide by the increased index.*

As particular examples we give

$$D^{-1}x^2 = \tfrac{1}{3}x^3 \qquad D^{-1}1/x^3 = D^{-1}x^{-3} = -\tfrac{1}{2}x^{-2} = -\tfrac{1}{2}\frac{1}{x^2}.$$
$$D^{-1}\sqrt{x} = D^{-1}x^{\frac{1}{2}} = \tfrac{2}{3}x^{\frac{3}{2}} \qquad D^{-1}x^0 = x$$

**96. Integral of a sum of two functions.**

We have to show that

$$D^{-1}(u + v) = D^{-1}u + D^{-1}v$$

Now we know that $\quad DD^{-1}(u + v) = u + v$

and that $\quad D(D^{-1}u + D^{-1}v) = u + v$

therefore $\quad DD^{-1}(u + v) = D(D^{-1}u + D^{-1}v)$

It follows by operating upon both sides with $D^{-1}$ that

$$D^{-1}(u + v) = D^{-1}u + D^{-1}v$$

The extension of the process to the sum of a finite number of integrations is easily made.

**97. Integral of the product of a constant and a function.**

We have to show that $\quad D^{-1}Cu = CD^{-1}u$

Now $\quad DD^{-1}Cu = Cu$

and $\quad DCD^{-1}u = CDD^{-1}u = Cu$

Hence $\quad DD^{-1}Cu = DCD^{-1}u$

It follows by operating upon both sides with $D^{-1}$ that

$$D^{-1}Cu = CD^{-1}u$$

**98. Integration of a polynomial.**

Let $f(x) = ax^n + bx^{n-1} + \ldots + hx + k$

By Arts. 96, 97, 95 we have

$$\begin{aligned} D^{-1}f(x) &= D^{-1}(ax^n + bx^{n-1} + \ldots + hx + k) \\ &= D^{-1}ax^n + D^{-1}bx^{n-1} + \ldots + D^{-1}hx + D^{-1}k \\ &= aD^{-1}x^n + bD^{-1}x^{n-1} + \ldots + hD^{-1}x + kD^{-1}x^0 \\ &= \frac{a}{n+1}x^{n+1} + \frac{b}{n}x^n + \ldots + \frac{h}{2}x^2 + kx + C \end{aligned}$$

**99. Integration of a general polynomial.**

Let $$\mathrm{P}(x) = ax^{\alpha} + bx^{\beta} + \ldots + kx^{\kappa}$$

where $\alpha, \beta, \gamma \ldots \kappa$ have any values except $-1$. Then

$$\mathrm{D}^{-1}\mathrm{P}(x) = \frac{a}{\alpha+1}x^{\alpha+1} + \frac{b}{\beta+1}x^{\beta+1} + \ldots + \frac{k}{\kappa+1}x^{\kappa+1} + \mathrm{C}$$

**100. Integration of a function of a function of x.**

No general statement of the solution of this problem can be made, as it is only a small number of compound functions that can be integrated. A few illustrations are given of the integration of certain compound algebraic functions.

Ex. 1. *To solve* $$\frac{dy}{dx} = \sqrt{(1-x)}$$

We change the variable so as to reduce the integrand to a power of the new variable. Writing $$z = 1 - x$$

$$\frac{dy}{dx} = z^{\frac{1}{2}}$$

But we must also change the independent variable from $x$ to $z$ on the left-hand side. This is effected by writing (see Art. 66)

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz}$$

We now have $$\frac{dy}{dz} = z^{\frac{1}{2}}(-1) = -z^{\frac{1}{2}}$$

Integrating with respect to $z$,

$$y = -\tfrac{2}{3}z^{\frac{3}{2}} + \mathrm{C} = -\tfrac{2}{3}(1-x)^{\frac{3}{2}} + \mathrm{C}$$

It may be noted that we have here a particular case of the equation

$$\frac{dy}{dx} = f'(ax+b)$$

the solution of which is $$y = \frac{1}{a}f(ax+b) + \mathrm{C}$$

Ex. 2. *To solve* $$\frac{dy}{dx} = \frac{x+1}{\sqrt{(x^2+2x+3)}}$$

In this we substitute $z = x^2 + 2x + 3$, and

$$\frac{dz}{dx} = 2(x+1)$$

Therefore $$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = \frac{dy}{dx}\Big/\frac{dz}{dx}$$

$$= \frac{x+1}{z^{\frac{1}{2}}}\cdot\frac{1}{2(x+1)} = \frac{1}{2}z^{-\frac{1}{2}}$$

whence $$y = z^{\frac{1}{2}} + \mathrm{C} = \sqrt{(x^2+2x+3)} + \mathrm{C}$$

This is a particular case of the equation

$$\frac{dy}{dx} = (ax + b)f'(ax^2 + 2bx + c)$$

the solution of which is

$$y = \tfrac{1}{2}f(ax^2 + 2bx + c) + C$$

**101. The logarithmic function.**

The integration of $x^n$ in the form

$$D^{-1}x^n = \frac{1}{n+1}x^{n+1} \quad (n \neq -1)$$

raises the question of the solution of the equation

$$\frac{dy}{dx} = \frac{1}{x}$$

The formula given in Art. 95 solves a set of equations which may be written

$$\frac{dy}{dx} = \ldots x^3, x^2, x^1, x^0, \quad , x^{-2}, x^{-3}, \ldots$$

this set exhibits a gap which it is our desire to fill. As we have solved

$$\frac{dy}{dx} = x^n$$

for all values of $n$, except $-1$, we might have arranged a series of equations which would have exhibited the want of continuity in the sequence of indices more completely, though perhaps not so strikingly. The important matter is that

$$\frac{dy}{dx} = \frac{1}{x}$$

is the only equation of the type which cannot be solved by a monomial in $x$.

The problem is solved by discovering a function whose differential coefficient answers the question proposed by the equation; the function sought is the logarithmic function, whose differential coefficient we shall now discuss. It will perhaps be of use to recall some of the properties of the function.

The definition of a logarithm gives the identity

$$a^{\log_a x} = x$$

where $a$ is the base. The base is, for ordinary calculation, taken as 10, the basis of our system of numeration, but another base is used in the infinitesimal calculus, which is the incommensurable number $e = 2{\cdot}7182818\ldots$, the limit of $(1 + x)^{1/x}$, when $x \to 0$. This number is discussed in Appendix II; it shares with $\pi$ the distinction of having

been studied more carefully by mathematicians than any other incommensurable number.

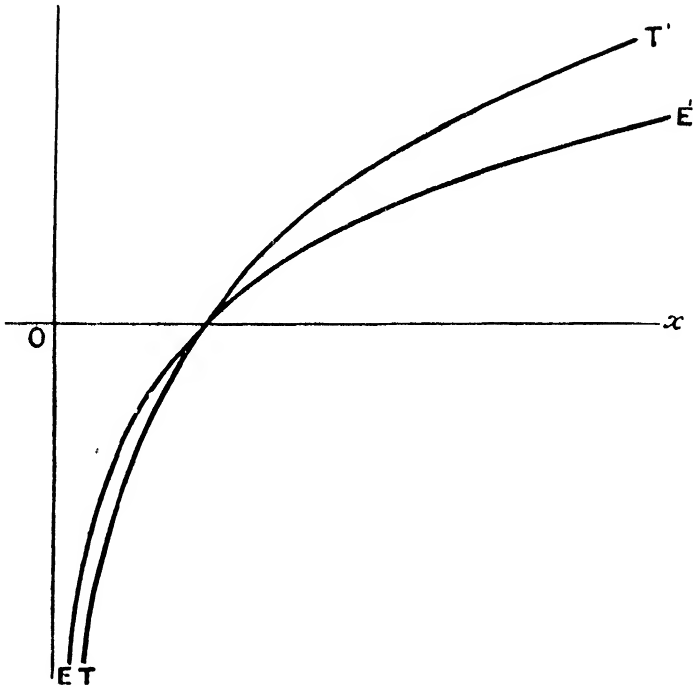



FIG. 43.

The diagram, Fig. 43, gives the graphs of $\log_e x$ and $\log_2 x$, which are respectively $EE'$ and $TT'$. The function $\log x$ is continuous in $[0, \infty]$, and its elementary properties are

$$\log_e x + \log_e y = \log_e xy$$
$$\log_e x - \log_e y = \log_e x/y$$
$$\log_e x^n = n \log_e x$$
$$\log_e x = \mu \log_{10} x$$
$$\mu = 2{\cdot}30258... \qquad \mu^{-1} = 0{\cdot}43429...$$

**102. Differential coefficient of $\log_e x$.**

Let
$$y = \log_a x$$
$$y + \delta y = \log_a(x + h)$$
then
$$\delta y = \log_a(x + h) - \log_a x = \log_a(1 + h/x)$$
$$\frac{\delta y}{\delta x} = \frac{1}{h}\log_a(1 + h/x) = \frac{1}{x}\log_a(1 + h/x)^{x/h}$$

Writing $z = h/x$, we have

$$\frac{\delta y}{\delta x} = \frac{1}{x}\log_a(1 + z)^{1/z}$$

Now let us give to $h$ a sequence of values whose limit is zero; with the restriction that $x + h$ is always positive, $z$ takes a sequence of values

$$z_1, \quad z_2, \quad z_3, \ldots$$

whose limit is zero, since $x$ must be finite.

But $\lim (1 + z)^{1/z} = e$, $z \to 0$; therefore

$$\frac{dy}{dx} = \frac{1}{x}\log_a e$$

since $\log_a x$ is continuous at $x = e$.

And if $y = \log_e x$, we have

$$\frac{dy}{dx} = \frac{d}{dx}\log_e x = \frac{1}{x}$$

It follows that $$\mathrm{D}^{-1}\frac{1}{x} = \log_e x$$

We shall in the future imply, when no base is expressed, that the base is $e$, and write always

$$\mathrm{D}^{-1}\frac{1}{x} = \log x$$

As a consequence of the work above,

$$\frac{d}{dx}\log_{10}x = \frac{1}{x}\log_{10}e = \frac{0{\cdot}43429\ldots}{x}$$

The gap in the series of functions alluded to in Art. 101 is now filled by $\log x$. It is the simplicity of the differential coefficient of this function which constitutes the real claim of $e$ to be used as a base in all theoretical developments.

**103. Integration of the reciprocal of the linear function.**

We can now solve the equation

$$\frac{dy}{dx} = \frac{1}{ax + b}$$

For, by substituting $z = ax + b$, we have

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = \frac{1}{z}\cdot\frac{1}{a}$$

hence $$y = \frac{1}{a}\log z + \mathrm{C} = \frac{1}{a}\log(ax + b) + \mathrm{C}$$

As another example, we take

$$\frac{dy}{dx} = \frac{ax + b}{ax^2 + 2bx + c}$$

The solution of this equation can be found by writing

$$z = ax^2 + 2bx + c$$

and we obtain $\qquad y = \frac{1}{2}\log(ax^2 + 2bx + c) + C$

**104. Integration of certain cases of R(x).**

It is not possible at this stage to solve the problem completely; at present we take the case of the integration of

$$R(x) = \frac{P_n(x)}{P_2(x)}$$

where $P_2(x)$ is a quadratic function which has real factors. Now, if $P_2(x) = (x - \alpha)(x - \beta)$ and the result of division gives a quotient $Q(x)$ and a remainder $ax + b$,

$$D^{-1}R(x) = D^{-1}\left[Q(x) + \frac{ax + b}{(x - \alpha)(x - \beta)}\right]$$

By the method of partial fractions

$$\frac{ax + b}{(x - \alpha)(x - \beta)} = \frac{A}{x - \alpha} + \frac{B}{x - \beta}$$

and $\qquad ax + b = A(x - \beta) + B(x - \alpha).$

Writing $x = \alpha, \beta$ successively,

$$A = \frac{a\alpha + b}{\alpha - \beta} \qquad B = \frac{a\beta + b}{\beta - \alpha}$$

Hence $\quad D^{-1}R(x) = D^{-1}Q(x) + AD^{-1}\dfrac{1}{x - \alpha} + BD^{-1}\dfrac{1}{x - \beta}$

$$= D^{-1}Q(x) + A\log(x - \alpha) + B\log(x - \beta)$$

where $D^{-1}Q(x)$ is known, as $Q(x)$ is a polynomial.

The case of $\alpha = \beta$ is solved by writing

$$\frac{ax + b}{(x - \alpha)^2} = \frac{a(x - \alpha) + a\alpha + b}{(x - \alpha)^2} = \frac{a}{x - \alpha} + \frac{a\alpha + b}{(x - \alpha)^2}$$

and $\quad D^{-1}\dfrac{ax + b}{(x - \alpha)^2} = aD^{-1}\dfrac{1}{x - \alpha} + (a\alpha + b)D^{-1}(x - \alpha)^{-2}$

$$= a\log(x - \alpha) - (a\alpha + b)(x - \alpha)^{-1}$$

The function $R(x)$ can also be integrated with our present knowledge even if the quadratic function cannot be resolved into linear factors, provided that, when

$$R(x) = Q(x) + \frac{ax + b}{px^2 + qx + r}$$

we have $a = 2kp$, $b = kq$; for in this case

$$D^{-1}\frac{ax + b}{px^2 + qx + r} = kD^{-1}\frac{2px + q}{px^2 + qx + r} = k\log(px^2 + qx + r)$$

Other cases require the use of functions whose differential coefficients have not yet been discussed.

**Illustrative examples.**

Ex. 1. *To integrate* $\dfrac{1}{x^2 - 1}$

Now $$\frac{1}{x^2 - 1} = \frac{1}{2}\frac{1}{x - 1} - \frac{1}{2}\frac{1}{x + 1}$$

Therefore $$D^{-1}\frac{1}{x^2 - 1} = \frac{1}{2}D^{-1}\frac{1}{x - 1} - \frac{1}{2}D^{-1}\frac{1}{x + 1}$$

$$= \frac{1}{2}\log(x - 1) - \frac{1}{2}\log(x + 1) = \frac{1}{2}\log\frac{x - 1}{x + 1}$$

Ex. 2. *To integrate* $\dfrac{(x - 1)(x + 2)}{x(x + 1)}$

Now $$\frac{(x - 1)(x + 2)}{x(x + 1)} = 1 - \frac{2}{x(x + 1)} = 1 - \frac{2}{x} + \frac{2}{x + 1}$$

$$D^{-1}\frac{(x - 1)(x + 2)}{x(x + 1)} = D^{-1}x^0 - 2D^{-1}\frac{1}{x} + 2D^{-1}\frac{1}{x + 1}$$

$$= x - 2\log x + 2\log(x + 1)$$

$$= x - 2\log\frac{x}{x + 1}$$

Ex. 3. *To integrate* $\dfrac{x^3}{(x - 1)^2}$

Now $$\frac{x^3}{(x - 1)^2} = x + 2 + \frac{3x - 2}{(x - 1)^2} = x + 2 + \frac{3}{x - 1} + (x - 1)^{-2}$$

and $$D^{-1}\frac{x^3}{(x - 1)^2} = \frac{1}{2}x^2 + 2x + 3\log(x - 1) - (x - 1)^{-1}$$

## 105. The value of $D^{-1}x^{-1}$, when x is negative.

In performing certain mathematical operations upon $x$ we have to remember that restrictions may be imposed upon $x$ by the nature of the operation. Thus, if $x$ is negative, it is impossible to take its square root; also $\log x$ is meaningless, unless $x$ is positive. Hence when we write

$$D^{-1}x^{-1} = \log x$$

we imply that $x$ is situated in the range $[0, \infty]$.

But an answer may be given to the question $D^{-1}x^{-1}$ even when $x$ is negative. For in this case

$$D^{-1}x^{-1} = \log(-x)$$

here, $x$ being negative, $-x$ is positive. That this result is true is shown by writing $-x = X$, and differentiating,

$$\frac{d}{dx}\log(-x) = \frac{d}{dX}\log X \cdot \frac{dX}{dx} = \frac{1}{X}\cdot(-1) = \frac{1}{x}$$

The whole answer to the question $D^{-1}x^{-1}$ is that, when $x$ lies in $[0, \infty]$,

$$D^{-1}x^{-1} = \log x$$

and when $x$ lies in $[-\infty, 0]$,

$$D^{-1}x^{-1} = \log(-x)$$

In the examples solved above we might have entered into greater detail. Thus, in writing

$$D^{-1}\frac{1}{x^2-1} = \frac{1}{2}\log\frac{x-1}{x+1}$$

we ought, perhaps, to have added that this holds when $x$ lies in either $[-\infty, -1]$ or $[1, \infty]$, and that

$$D^{-1}\frac{1}{x^2-1} = \frac{1}{2}\log\frac{1-x}{x+1}$$

when $x$ lies in $[-1, 1]$.

**106. Integration of $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\sin^2 x$, $\cos^2 x$.**

As a direct consequence of formulae of differentiation, we have

$$D^{-1}\sin x = -\cos x \qquad D^{-1}\cos x = \sin x$$

Again, since
$$\tan x = \frac{1}{\sec x}\tan x \sec x$$
$$= \frac{1}{\sec x}\frac{d}{dx}\sec x = \frac{d}{dx}\log\sec x$$

therefore
$$D^{-1}\tan x = \log\sec x$$

Also, we have
$$\sec x = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$$
$$= \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x}$$

and
$$D^{-1}\sec x = \log(\sec x + \tan x)$$

Similarly
$$D^{-1}\cot x = -\log\operatorname{cosec} x \;(= \log\sin x)$$
$$D^{-1}\operatorname{cosec} x = -\log(\operatorname{cosec} x + \cot x)$$

The integration of $\sin^m x$, $\cos^m x$, where $m$ is a positive integer, can be effected by expanding these functions in a series of sines or cosines of multiple angles. Thus, by Trigonometry, we know that

$$\begin{aligned}
\sin^2 x &= \tfrac{1}{2}(1 - \cos 2x)\\
\cos^2 x &= \tfrac{1}{2}(1 + \cos 2x)\\
\sin^3 x &= \tfrac{1}{4}(3\sin x - \sin 3x)\\
\cos^3 x &= \tfrac{1}{4}(3\cos x + \cos 3x)\\
\sin^4 x &= \tfrac{1}{8}(3 - 4\cos 2x + \cos 4x)\\
\cos^4 x &= \tfrac{1}{8}(3 + 4\cos 2x + \cos 4x)
\end{aligned}$$

whence it follows that

$$\begin{aligned}
D^{-1}\sin^2 x &= \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x)\\
D^{-1}\cos^2 x &= \tfrac{1}{2}(x + \tfrac{1}{2}\sin 2x)\\
D^{-1}\sin^3 x &= \tfrac{1}{4}(-3\cos x + \tfrac{1}{3}\cos 3x)\\
D^{-1}\cos^3 x &= \tfrac{1}{4}(3\sin x + \tfrac{1}{3}\sin 3x)\\
D^{-1}\sin^4 x &= \tfrac{1}{8}(3x - 2\sin 2x + \tfrac{1}{4}\sin 4x)\\
D^{-1}\cos^4 x &= \tfrac{1}{8}(3x + 2\sin 2x + \tfrac{1}{4}\sin 4x)
\end{aligned}$$

## EXERCISES IX

Find the integrals of the following functions

1. $1 + x + x^2$
2. $(1 + x)(x - 2)$
3. $(1 - x)^3$
4. $1 - x^{-2} - x^{-3}$
5. $(a^2 - x^2)^2$
6. $(1 - x)(1 + x^2)$
7. $ax + bx^3$
8. $(ax + b)(cx + d)$
9. $x^p + x^{-p}$
10. $x^2 + 1/\sqrt{x}$
11. $(x + 1)/\sqrt{x}$
12. $(x^2 + x + 1)/x^4$
13. $\dfrac{x^3 - 1}{x}$
14. $\dfrac{x^3}{x - 1}$
15. $\dfrac{1}{1 - 2x}$
16. $\dfrac{2x - 1}{2x + 3}$
17. $\dfrac{x + 1}{x^2 - 3x + 2}$
18. $\dfrac{1}{4 - x^2}$
19. $\dfrac{1}{x^2 - 4x + 3}$
20. $\dfrac{x^2 + x}{x^2 - x + 1}$
21. $\dfrac{x}{x^4 - a^4}$
22. $\dfrac{x}{(x^2 + a^2)^2}$
23. $\dfrac{x^3 - 1}{x + 1}$
24. $\dfrac{3x}{2x^2 - x - 1}$
25. $\dfrac{13}{6x^2 + 5x - 6}$
26. $\dfrac{3x + 1}{x(x^2 - 1)}$
27. $\dfrac{16}{x^2(x + 4)}$

Evaluate

28. $D^{-1}\sqrt[3]{x}$
29. $D^{-1}1/x^3$
30. $D^{-1}\sqrt{(2x + 1)}$
31. $D^{-1}(2x + 1)^{-2}$
32. $D^{-1}(ax + b)^n$
33. $D^{-1}(x - 1/x)^2$
34. $D^{-1}(x^2 + x + 1)^2$
35. $D^{-1}(3x - 1)(x + 1)$
36. $D^{-1}(x + 1)^2/x$

37. $D^{-1}x^3/(x+1)$

38. $D^{-1}(x^2-3)(x^2+1)x^{-2}$

39. $D^{-1}(x-1)^2/\sqrt{x}$

40. $D^{-1}\dfrac{1}{x(x-1)}$

41. $D^{-1}\dfrac{x-3}{x(x-1)}$

42. $D^{-1}\dfrac{x^2-2}{x(x-1)}$

43. $D^{-1}\dfrac{x^2+1}{x^3+3x-5}$

44. $D^{-1}\dfrac{x^2-x}{x^2+x+1}$

45. $D^{-1}\dfrac{x^2-1}{x^4+x^2+1}$

46. $D^{-1}\dfrac{(x+1)^2}{x^2}$

47. $D^{-1}\dfrac{x^2}{(x+1)^2}$

48. $D^{-1}\dfrac{20x+28}{(2x-3)^2}$

49. $D^{-1}\dfrac{1}{x(x+1)(x+2)}$

50. $D^{-1}\dfrac{1+4x^2}{x(2x-1)^2}$

51. $D^{-1}\dfrac{x^2}{x^3-1}$

52. $D^{-1}\dfrac{x+1}{x^3-1}$

53. $D^{-1}\dfrac{3x+2}{x^3-x}$

54. $D^{-1}\dfrac{x^2}{(x-1)^2(x+1)}$

55. $D^{-1}\dfrac{x^6}{(x-1)^3}$

56. $D^{-1}x\sqrt{(a^2+x^2)}$

57. $D^{-1}\dfrac{x}{\sqrt{(a^2+x^2)}}$

58. $D^{-1}x^{n-1}(a^n+x^n)^p$

59. $D^{-1}\dfrac{1}{x}\log x$

60. $D^{-1}\dfrac{1}{x\log x}$

61. $D^{-1}\sin 3x$

62. $D^{-1}36\sin^3 3x$

63. $D^{-1}(\sin x+\cos x)^2$

64. $D^{-1}\dfrac{\cos x}{1+\sin x}$

65. $D^{-1}\dfrac{1+\sin x}{\cos x}$

66. $D^{-1}\tan^2 x$

67. $D^{-1}\tan^4 x$

68. $D^{-1}\tan^3 x$

69. $D^{-1}\sec^4 x$

70. $D^{-1}\sec x\tan^3 x$

71. $D^{-1}\dfrac{1}{1+\tan x}$

72. $D^{-1}\dfrac{a\sin x+b\cos x}{c\sin x+d\cos x}$

73. $D^{-1}\sin mx\cos nx$

74. $D^{-1}\sin mx\sin nx$

75. $D^{-1}\cos mx\cos nx$

# CHAPTER X

## AREAS. VOLUMES

### 107. First application of integration.

The main object of this chapter is to apply integration to problems of mensuration. But before turning to it, a simple problem of kinematics is taken; this course is adopted because the language of mechanics can be interpreted so directly by the symbols of the infinitesimal calculus.

It has been shown in Art. 88 that, if the displacement ($x$) of a point moving along $Ox$ is known at every instant ($t$), the velocity ($v$) is deduced by differentiating $x$ with respect to $t$, and that by differentiating $v$ the acceleration ($f$) is determined. We shall now show when $f$ is constant, (i) how $v$ may be found and (ii) how $x$ may be determined from the value $v$; that is, we find *the position at any time of a particle moving along $Ox$ with constant acceleration, when its initial velocity and position are given.*

With the notation of Art. 88,

$$\frac{dv}{dt} = f$$

Since $f$ is constant, we have on integration

$$v = ft + C_1$$

Again, if $v = V$, when $t = 0$, we have, $C_1 = V$, and

$$v = ft + V \qquad \frac{dx}{dt} = ft + V$$

Integrating this, $\quad x = \frac{1}{2}ft^2 + Vt + C_2$

Here, supposing that $x = 0$, when $t = 0$, we obtain

$$C_2 = 0$$

and the well-known formula is found,

$$x = \frac{1}{2}ft^2 + Vt$$

We could, by taking $f$ as a polynomial function of $t$, work out other cases, but such examples are not worth discussion, as they do not occur naturally. Other problems which are of importance will be

discussed at a later stage. The problem given allows the student to form a conception of the application of integration to problems in kinematics; indeed, all such problems are reduced by laws based upon experiment to relations connecting acceleration, velocity and position, the further reduction of which is effected by the infinitesimal calculus.

**108. Differential coefficient of a trapezoidal area A(x).**

Let $A(x)$ be an area bounded by the axes $Ox$, $Oy$, a curve whose equation is $y = \varphi(x)$ and a variable ordinate defined by the abscissa $x$, we proceed to determine

$$\frac{dA}{dx}$$

It will be assumed in the first instance that $y = \varphi(x)$ lies wholly above $Ox$.

Let $\qquad A(x) = OMPB \qquad x = OM$

then $\qquad A(x) + \delta A(x) = OM'P'B$

$$\delta A(x) = PMM'P'$$

Now it is clear (see Fig. 44) that there is some point $Q$ in $PP'$ such that, if $RQR'$ is drawn parallel to $Ox$ to meet the ordinates at $P$ and $P'$ (either being produced, if necessary) in $R$ and $R'$, then the rectangle $RMM'R'$ is equal to the area $PMM'P'$; hence

$$\delta A(x) = \text{rect. } QN \,.\, MM'$$

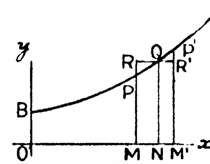


FIG. 44.

Now let $ON = x + \theta\,\delta x$, where $\theta$ is a proper fraction. Then we have

$$\delta A(x) = \delta x\ \varphi(x + \theta\,\delta x)$$

$$\frac{\delta A(x)}{\delta x} = \varphi(x + \theta\,\delta x)$$

and the relation holds even when $\delta x$ is negative. Proceeding to make $\delta x \to 0$, $\varphi(x + \theta\,\delta x) \to \varphi(x)$, and we have

$$\frac{dA}{dx} = \varphi(x)$$

It is assumed in this proof that the curve $y = \varphi(x)$ is continuous.

**109. Area bounded by a curve, the axes and an ordinate.**

The restrictions imposed in Art. 108 hold in the case of the curved boundary of this area. We have

$$\frac{dA}{dx} = \varphi(x)$$

If we can find a function which when differentiated gives $\varphi(x)$, we have a solution. The constant in the solution has to be determined; this is found by noticing that

$$A(0) = 0$$

**110. Integraph.**

The solution of the problem in Art. 109 is effected practically by an instrument called the Integraph, the principle of which is directly deduced from considerations already proved.

Let $BP$ be the curve $y = \varphi(x)$, and let $MK$ be measured to the left along $Ox$ from $M$, the foot of the ordinate of $P$, and be of unit length; also let $Mp$ be equal to the area $OMPB$ [ $= \mathrm{A}(x)$]. Then

$$\tan MKP = MP/KM = MP = \varphi(x)$$

Again, $\tan Mtp = \frac{d\mathrm{A}}{dx} = MP = \varphi(x)$

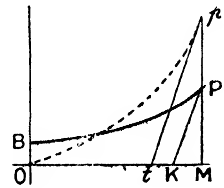


FIG. 45.

Therefore, $KP$ is parallel to the tangent at $p$.

The Integraph is designed with two pointers $P$ and $p$ so controlled that as $P$ describes $BP$, $p$ moves so that the tangent to its path is parallel to $KP$. In such an instrument the reading given by $Mp$ in any position is the area $OMPB$, provided that the pointer starts at $O$, and therefore $\mathrm{A}(0) = 0$.

**111. Illustrative examples.**

Ex. 1. *To find the area of a right-angled triangle.*

Let $OBC$ be the triangle; take the axis $Ox$ along $OB$ and consider a point $P$ on $OC$.

The equation of $OC$ is $y = mx$ and $\mathrm{A}(x)$ is the area $OMP$.

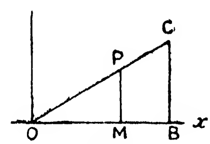


FIG. 46.

Then $\frac{d\mathrm{A}}{dx} = MP = mx$

$$\mathrm{A}(x) = \tfrac{1}{2}mx^2 + C_1$$

and since $\mathrm{A}(0) = 0 \quad C_1 = 0$

whence $\mathrm{A}(x) = \tfrac{1}{2}mx^2$

The area $OBC = \mathrm{A}(b) = \tfrac{1}{2}mb^2 = \tfrac{1}{2}mb \,.\, b = \tfrac{1}{2}OB \,.\, BC$.

Ex. 2. *To find the area of a trapezium.*

Let $Ox$ be perpendicular to the parallel sides, and let the equation of $BC$ be $y = mx + b$. Then, taking $\mathrm{A}(x)$ as the area $OMPB$, we have

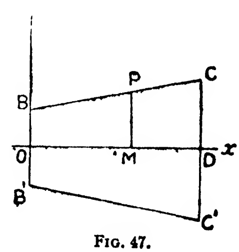


FIG. 47.

$$\frac{d\mathrm{A}}{dx} = mx + b$$

$$\mathrm{A}(x) = \tfrac{1}{2}mx^2 + bx + C_1$$

Also $C_1 = 0$, because $\mathrm{A}(0) = 0$, and

$\mathrm{A}(x) = \tfrac{1}{2}x(mx + b + b) = \tfrac{1}{2}OM(BO + PM)$.

Thus, the area $OBCD = \tfrac{1}{2}OD(BO + CD)$.

Again, the area $OB'C'D = \tfrac{1}{2}OD\,(OB' + DC')$.

Therefore the area of the trapezium $= \tfrac{1}{2}OD(BB' + CC')$.

Ex. 3. *To find the area of the part of the curve $y = x^2(3 - x)$ which lies in the first quadrant.*

We require the area $OAPO$, but we must solve the more general problem of finding the area $OMP = \mathrm{A}(x)$.

Now $$\frac{d\mathrm{A}}{dx} = MP = 3x^2 - x^3$$

$$\mathrm{A}(x) = x^3 - \tfrac{1}{4}x^4$$

no constant being added, because $\mathrm{A}(0) = 0$.

The area required $= \mathrm{A}(3) = 27(1 - \tfrac{3}{4}) = 6\tfrac{3}{4}$.

This answer may be checked by drawing the curve on squared paper and estimating from the diagram the number of squares and portions of squares included in the area.

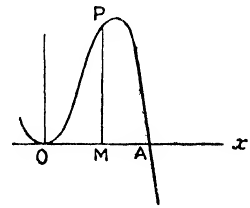



FIG. 48.

The examples just given illustrate the application of integration, (i) to two problems whose results are well known, and (ii) to a problem which might be insoluble by elementary methods. This is characteristic of the power of the method.

It may be noticed, too, that, whenever we know the solution of an equation
$$\frac{dy}{dx} = \varphi(x)$$
we have the solution of a class of problems concerned with the area bounded by $y = \varphi(x)$, the axes and an ordinate.

Further, in each of the examples a far more general problem is solved than the single particular instance presented for solution; this was especially the case in Ex. 3. Indeed, the solution of
$$\frac{d\mathrm{A}}{dx} = \varphi(x)$$
when we know it, solves a second class of problems in which the constant C has different values. For, if we take $\mathrm{A}(x)$ as the area (supposed to be above $Ox$) lying between $y = \varphi(x)$, $Ox$ and two ordinates, one of which is fixed, we have still
$$\frac{d\mathrm{A}}{dx} = \varphi(x)$$
and by integration we obtain $\mathrm{A}(x)$. Thus, in the diagram, if the area $BCC'B'$ is required, we take $$\mathrm{A}(x) = \text{area } BCMP.$$

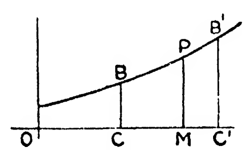



FIG. 49.

In the solution of this problem we determine the constant by writing $\mathrm{A}(b) = 0$; the required area is then $\mathrm{A}(b')$, where $b$ and $b'$ are the abscissae of $B$ and $B'$ respectively.

Ex. 4. *To find the area of*
$$y = (x + 2)(1 - x)$$
*cut off by $Ox$.*

The parabola cuts $Ox$ at $A\,(1, 0)$ and $B\,(-2, 0)$. Let the area $BMP$ be $\mathrm{A}(x)$, then

$$\frac{d\mathrm{A}}{dx} = y = 2 - x - x^2$$

$$\mathrm{A}(x) = 2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \mathrm{C}$$

Now $\mathrm{A}(-2) = 0$, therefore $\mathrm{C} = 3\tfrac{1}{3}$, and

$$\mathrm{A}(x) = 2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + 3\tfrac{1}{3}$$

Hence $\mathrm{A}(1)$, the required area, is $4\tfrac{1}{2}$.

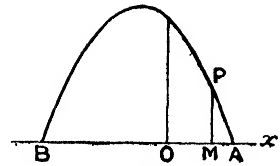



FIG. 50.

**112. Extended meaning of A(x).**

In Art. 108 we found $\mathrm{A}(x)$ on the assumption that $y = \varphi(x)$ was wholly above $Ox$; we must now discuss the new meaning which may be given to $\mathrm{A}(x)$ in the case in which $y = \varphi(x)$ crosses $Ox$.

If we regard $\varphi(x)$ as the derived function of $\mathrm{A}(x)$, it is clear from Art. 42 that so long as $\varphi(x)$ is positive, $\mathrm{A}(x)$ increases, but that, when $\varphi(x)$ is negative, $\mathrm{A}(x)$ decreases.

Again, referring to the integraph curve (Fig. 51), in which $Cc$ is equal to the area $OCB$, we see that after passing $C$, the ordinate

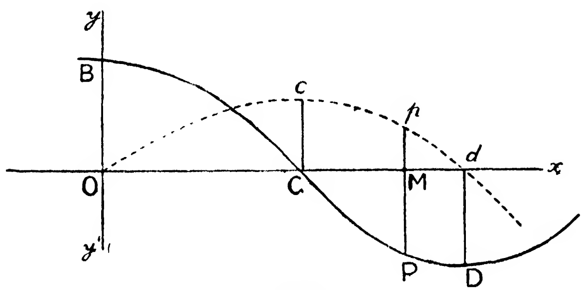



FIG. 51.

of the curve, $y = \varphi(x)$ becomes negative, the pointer of the integraph which describes the graph of $\mathrm{A}(x)$ approaches $Ox$. To estimate the fall, let us for a moment regard $y = \varphi(x)$ in the sense in which $Oy'$ is the positive direction; in this case, the fall becomes a rise and the rise from $c$ to $p$ is equal to the area $CPM$. Returning now to the usual conventions, it follows that

$$\begin{aligned} Mp &= Cc - (Cc - Mp) \\ &= \text{area } OCB - \text{area } CPM \end{aligned}$$

Also looking at the diagram in which $y = \mathrm{A}(x)$ meets $Ox$ at $d$, we see that the area $OCB$ = the area $CDd$.

Thus if $\mathrm{A}(x)$ is the solution of

$$\frac{d\mathrm{A}}{dx} = \varphi(x)$$

even though $y = \varphi(x)$ may cut $Ox$, we have a meaning for $\mathrm{A}(x)$;

for, denoting the trapezoidal areas given by $y = \varphi(x)$, and $Ox$ by $A_1, A_2, A_3, \ldots$, of which $A_1$ is above $Ox$, $A_2$ is below, and so on, we have

$$\mathrm{A}(x) = A_1 - A_2 + A_3 - \ldots$$

The reader may have noticed that the areas have been throughout lettered in the order in which they would be described in a counter clockwise circuit. Just as an athlete runs round a track with his left foot always inside, so we have suggested a definite direction of description. Again, so long as the area is above $Ox$, our convention has agreed with another convention made in

$$\frac{d\mathrm{A}(x)}{dx} = \varphi(x)$$

namely, that the part of the axis of $x$ included in the circuit is also described in the direction of $x$ increasing. Now both of these

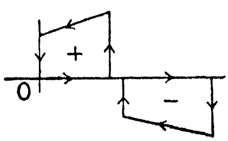



FIG. 52$a$.

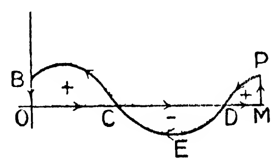



FIG. 52$b$.

requirements are satisfied if we suppose that the boundary of $\mathrm{A}(x)$ is described so that the axis of $x$ is described in the direction $Ox$ and that the contribution to $\mathrm{A}(x)$ is positive when the area described lies on the left, and negative when the area is on the right. The first figure (Fig. 52$a$) illustrates the two cases separately; in the second (Fig. 52$b$) the two cases are combined. The function $\mathrm{A}(x)$ in Fig. 52$b$ is then

the area $OCMPDECBO = OCB - CDE + DMP$

**113. New interpretation of the relation between f(x) and f′(x).**

With the extended interpretation of $\mathrm{A}(x)$ we have a complete interpretation of the relation between the functions $\mathrm{A}(x)$, $\varphi(x)$, which are connected by the equation

$$\frac{d\mathrm{A}(x)}{dx} = \varphi(x)$$

But we have already discussed the same relation with different symbols in the form

$$\frac{df(x)}{dx} = f'(x)$$

The two problems may be considered together by writing

$$\mathrm{A}(x) = f(x) \qquad \varphi(x) = f'(x)$$

and a new interpretation of the results of Art. 46 obtained by considering $f'(x)$ as the original graph and $f(x)$ as a function which expresses the area (in its extended meaning) lying between $Ox$, the curve $y = f'(x)$ and the ordinate given by the abscissa $x$

**114. Volumes of solids of revolution.**

Let $V(x)$ be the volume generated by the revolution about $Ox$ of the area which in Art. 108 was denoted by $A(x)$. Then, referring to Fig. 44 and taking $V(x) + \delta V(x)$ as the volume generated by $OM'P'B$, we have

$$\begin{aligned}\delta V(x) &= \text{volume generated by } PMM'P' \\ &= \pi\, MM' \,.\, QN^2\end{aligned}$$

where $Q$ is some point of $PP'$, but not in general the same as the point in Art. 108.

Hence $$\frac{\delta V}{\delta x} = \pi QN^2 = \pi[\varphi(x + \theta\,\delta x)]^2$$

where $\theta$ is a proper fraction. The value of $\theta$ varies with the length of $\delta x$, but is always between 0 and 1. Proceeding to the limit in which $\delta x \to 0$, we have

$$\frac{dV}{dx} = \pi[\varphi(x)]^2$$

The problem is simpler than the problem of areas, inasmuch as $[\varphi(x)]^2$ is essentially positive and $V(x)$ does not decrease even when $y = \varphi(x)$ crosses the axis. The extended meaning when the axis is crossed being that $V(x)$ is the sum of the volumes generated.

The student should notice that the integraph can be applied to this problem. For, if we write

$$\frac{d}{dx}(V/\pi) = [\varphi(x)]^2 = \chi(x)$$

and trace the graph of $\chi(x)$ on the paper, then the integraph when applied to $y = \chi(x)$ gives us the quantity $V/\pi$.

We may notice that the solution of

$$\frac{dy}{dx} = f(x)$$

when $f(x)$ is positive, provides us with the solution of two problems, (i) the area bounded by the graph of $f(x)$, two ordinates and $Ox$, and (ii) the fraction 0·31831... of the volume generated by the revolution round $Ox$ of the area bounded by the graph of $[f(x)]^{\frac{1}{2}}$, two ordinates and $Ox$.

**115. Illustrative examples.**

Ex. 1. *To find the volume of the frustum of a cone.*

The diagram is a section through the axis of the cone, which is $Ox$. The equation of the upper generator is

$$y = r + (R - r)x/h \quad \text{(Fig. 53)}$$

where $x = 0$, $x = h$ give the ends of the frustum

$$\frac{dV}{dx} = \pi y^2 = \pi\left[r + \frac{R - r}{h}x\right]^2$$

$$V(x) = \tfrac{1}{3}\pi\frac{h}{R - r}\left[r + \frac{R - r}{h}x\right]^3 + C$$

Now $V(0) = 0$; therefore

$$C = -\tfrac{1}{3}\pi\frac{hr^3}{R - r}$$

Whence $$V(x) = \tfrac{1}{3}\pi\frac{h}{R - r}\left[r + \frac{R - r}{h}x\right]^3 - \tfrac{1}{3}\pi\frac{hr^3}{R - r}$$

And the volume of the frustum $= V(h)$

$$= \tfrac{1}{3}\pi\frac{h}{R - r}(R^3 - r^3) = \tfrac{1}{3}\pi h(R^2 + Rr + r^2)$$

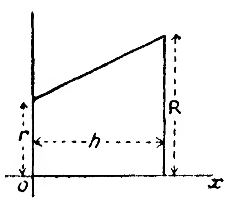



FIG. 53.

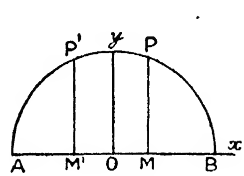



FIG. 54.

Ex. 2. *To find the volume of the frustum of a sphere.*

Let the equation of the circle which generates the sphere be

$$y^2 = r^2 - x^2$$

Then $$\frac{dV}{dx} = \pi y^2 = \pi(r^2 - x^2)$$

$$V(x) = \pi(r^2x - \tfrac{1}{3}x^3) + C \quad \text{(Fig. 54)}$$

Now, if $V(x)$ is the volume generated by $AMP$, then when $x = -r$, $V(x) = 0$, whence $C = \tfrac{2}{3}\pi r^3$,

$$V(x) = \pi(r^2x - \tfrac{1}{3}x^3 + \tfrac{2}{3}r^3)$$
$$= \tfrac{1}{3}\pi(r + x)(2r^2 + rx - x^2)$$

As verifications we have the volume of the hemisphere obtained by writing $x = 0$, and that of the sphere by writing $x = r$. The volume of the frustum generated by $MBP$ is equal to the symmetrical frustum generated by $AM'P'$, which is obtained by writing $-x$ for $x$.

Ex. 3. *To find the volume of a paraboloid.*

The generating parabola is $y^2 = 4ax$.

We have
$$\frac{dV}{dx} = \pi y^2 = 4\pi ax$$
$$V(x) = 2\pi ax^2$$
no constant being added, because $V(0) = 0$. The answer is usually expressed by saying that the volume of the paraboloid is one-half the volume of the cylinder upon the same base and of the same altitude.

**116. Approximative equation of a curve given by three points.**

A single observation of a physical quantity which is a function of $x$ gives one point upon its graph; $n$ observations provide us with $n$ points. Now, from the points available it is often possible to conjecture the shape of the graph in the neighbourhood of these points; the more points there are, the more plausible is the guess. If we have only two points, we assume a linear graph and join the points by a straight line; if there are three points, we assume that the function is quadratic and draw a parabola through the given points, having its axis perpendicular to the axis of $x$.

Let us take the case of three observations and suppose that they are equidistant; then, if we choose our axes properly, we may suppose that the three points upon the graph are
$$(-h, y_1) \qquad (0, y_2) \qquad (h, y_3)$$
Assuming that the function is
$$y = a + bx + cx^2$$
we have

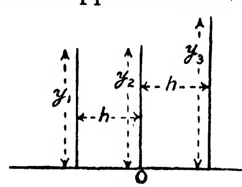



FIG. 55

$$y_1 = a - bh + ch^2 \qquad y_2 = a \qquad y_3 = a + bh + ch^2$$
It follows that
$$y_1 + y_3 = 2(a + ch^2) \qquad y_3 - y_1 = 2bh$$
and that
$$a = y_2 \qquad b = \frac{y_3 - y_1}{2h} \qquad c = \frac{y_1 - 2y_2 + y_3}{2h^2}$$
The form of the function is given by
$$y = y_2 + \frac{y_3 - y_1}{2h}x + \frac{y_1 - 2y_2 + y_3}{2h^2}x^2$$

The reader must understand that the assumption of a quadratic function as the form of the function is only one out of many which could be made. Its artificiality can be illustrated by taking a second form which will be useful afterwards, namely,
$$y^2 = a + bx + cx^2$$

Here, the geometrical assumption is that the curve which is the graph of the function is a central conic, having one of its principal axes upon $Ox$ and passing through the three points. Introducing the same data into this equation, we have, by an almost identical series of algebraical steps,

$$y^2 = y_2^2 + \frac{y_3^2 - y_1^2}{2h} x + \frac{y_1^2 - 2y_2^2 + y_3^2}{2h^2} x^2$$

The two functions thus found are both useful : the first will be used to find an approximative expression for the area bounded by the graph, $Ox$ and the extreme ordinates, and the second to find an approximation to the volume generated by the revolution of this trapezoidal area about $Ox$.

**117. Simpson's rule for a trapezoidal area.**

If we take an area bounded by a curve, two ordinates and the axis, Simpson's rule provides an approximative formula for this area when the extreme ordinates, the intermediate ordinate and the width of the area are given, provided the curve lies on one side of $Ox$ and passes continuously from one end to the other.

For if we indicate the three ordinates as $y_1$, $y_2$, $y_3$, in their order of sequence, we may assume that the curve is a parabola through the three points with its axis parallel to $Oy$ and write as an approximate form of the curve the equation found in Art. 116,

$$y = y_2 + \frac{y_3 - y_1}{2h} x + \frac{y_1 - 2y_2 + y_3}{2h^2} x^2$$

Then, since

$$\frac{dA}{dx} = y = y_2 + \frac{y_3 - y_1}{2h} x + \frac{y_1 - 2y_2 + y_3}{2h^2} x^2$$

$$A = y_2 x + \frac{y_3 - y_1}{4h} x^2 + \frac{y_1 - 2y_2 + y_3}{6h^2} x^3 + C$$

Also, remembering that $A(-h) = 0$,

$$C = y_2 h - \frac{y_3 - y_1}{4h} h^2 + \frac{y_1 - 2y_2 + y_3}{6h^2} h^3$$

and the area enclosed by the extreme ordinates $= A(h)$

$$= y_2 h + \frac{y_3 - y_1}{4h} h^2 + \frac{y_1 - 2y_2 + y_3}{6h^2} h^3 + C$$

$$= 2y_2 h + \tfrac{1}{3}(y_1 - 2y_2 + y_3)h$$

$$= \tfrac{1}{3}(y_1 + 4y_2 + y_3)h$$

A couple of simple tests of cases in which the formula is exact may be given ; they will perhaps even suffice to enable the student to

write down the formula, if he recollects its general form. Thus, in the rectangle $y_1 = y_2 = y_3$ and the area $= 2y_1h$; in the trapezium $2y_2 = y_1 + y_3$, and the area $= 2y_2h$. The formula gives, of course, an exact result also in the case of the parabola.

**118. Extension of Simpson's rule.**

It is not difficult to extend the formula to meet the cases in which more than three equidistant ordinates are given, and also when the ordinates are not equidistant. But it is usually sufficient to tackle such problems by successive applications of Simpson's rule as stated above.

Thus, if seven equidistant ordinates are given, the area is divided into three sub-areas by $y_3$ and $y_5$, and Simpson's rule is applied to them. The approximation to the area obtained is

$$\tfrac{1}{3}(y_1 + 4y_2 + y_3 + y_3 + 4y_4 + y_5 + y_5 + 4y_6 + y_7)h$$
$$= \tfrac{1}{3}[y_1 + y_7 + 4(y_2 + y_4 + y_6) + 2(y_3 + y_5)]h$$

from which a rule is at once contrived to meet the case when the number of given ordinates is odd; when this number is even, it is best to take the area bounded by the first or last pair separately, and to treat the remaining part of the area by the above method.

**119. Simpson's rule for volumes of revolution.**

We suppose as before that in the generating area three equidistant ordinates are given, and we assume that the generating curve has for its equation

$$y^2 = y_2^2 + \frac{y_3^2 - y_1^2}{2h}x + \frac{y_1^2 - 2y_2^2 + y_3^2}{2h^2}x^2$$

and solve
$$\frac{dV}{dx} = \pi y^2$$

Proceeding as in the case of the trapezoidal area,

$$V = \tfrac{1}{3}\pi(y_1^2 + 4y_2^2 + y_3^2)h$$

Simple tests corresponding to those given in Art 117 are numerous: the cylinder in which $y_1 = y_2 = y_3 = r$ and whose volume is $\pi r^2 . 2h$; the cone in which $y_1 = 0$, $2y_2 = y_3 = r$ and whose volume is $\frac{1}{3}\pi r^2 . 2h$; the sphere in which $y_1 = y_3 = 0$, $y_2 = r = h$, whose volume $= \frac{4}{3}\pi r^3$; and the frustum of the cone in which $y_1 = r$, $y_3 = R$, $y_2 = \frac{1}{2}(r + R)$, whose volume $= \frac{1}{3}\pi(r^2 + rR + R^2)2h$.

These results are accurate because the equation of the generating curve is in all cases expressible in the form

$$y^2 = a + bx + cx^2$$

**120. Trapezoidal areas whose bases rest upon Oy.**

We now discuss the area between a curve which does not cross $Oy$, the axis of $y$ and two lines parallel to $Ox$.

The area required is indicated in the diagram; it is the area $B_1BCC_1$. The solution may be obtained from the previous work, for we have

$$\text{area } B_1BCC_1 = \text{rect. } OC - \text{rect. } OB - \text{area } BB'C'C$$

The same problem is solved directly if we can express the equation of the curve in the form

$$x = \chi(y)$$

For then, if we write the area $B_1BPP_1$ as $A_1(y)$, we have, just as in Art. 108,

$$\frac{dA_1(y)}{dy} = \chi(y)$$

and by integration we can obtain $A_1(y)$.

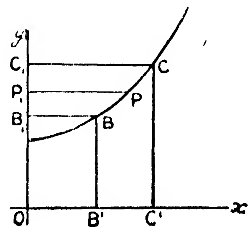



FIG. 56.

Volumes generated by the revolution of $B_1BCC_1$ about $Oy$ are determined by solving the equation

$$\frac{dV_1}{dy} = \pi[\chi(y)]^2$$

**121. Areas of certain oval curves and the volumes generated by their revolution.**

Consider the curve whose equation is

$$y = f(x) \pm g(x)$$

where $g(b) = 0, g(c) = 0$ and $g(x)$ is defined only for a range $(b, c)$ of $x$. The dotted line $y = f(x)$ on the figure bisects the chords of the oval which are parallel to $Oy$.

Let $PQ$ be one of these chords, so that

$$y_1 = MP = f(x) + g(x)$$
$$y_2 = MQ = f(x) - g(x)$$

and let $A(x)$ be the area $BQPB$; then we have that $\delta A(x)$ is the area $PQQ'P' = QP \,.\, \delta x$ nearly. Proceeding, we find

$$\frac{dA}{dx} = QP = MP - MQ = 2g(x)$$

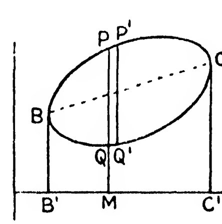



FIG. 57.

By integration $A(x)$ is found, the constant is determined by writing $A(b) = 0$, and the whole area is then given by $A(c)$.

The application of this method to examples leads generally to problems of integration of which we have not at present treated.

The volumes generated by the revolution of oval curves of this type which do not meet $Ox$ may be obtained in the same way. If

$V(x)$ is the volume generated by the revolution of $A(x)$ about $Ox$, we find

$$\frac{dV}{dx} = \pi(y_1^2 - y_2^2) = 4\pi f(x) \, . \, g(x)$$

The integraph is applied very simply to such problems as the determination of the oval area by allowing the pointer to travel round the whole circumference, $BQCPB$. For the area $BB'C'CQ$ will be recorded as the abscissa increases, while $CPBB'C'$ with a negative sign is recorded as the abscissa decreases, the pointer moving from $C$ to $P$ to $B$. Then the total area recorded will be the difference between the numerical values of the two trapezoidal areas, which is the area of the oval $BQCP$.

## EXERCISES X (A)

### Areas

1. Show (i) that the area bounded by $y = x^2$, $x = 1$, $y = 0$ is $\frac{1}{3}$, and (ii) that the area bounded by $y^2 = x^3$, $x = 1$, $y = 0$ is $\frac{2}{5}$.

2. Prove that the axis of $x$ cuts off from $y = 2x - 3x^2$ an area equal to $\frac{4}{27}$.

3. Show that the area of $y = (x + 1)(x - 2)$ cut off by $Ox$ is $4\frac{1}{2}$.

4. Find the area of the segment of $y^2 = 4ax$ cut off by $x = a$, and prove that it is bisected by $x = \frac{1}{2}a\sqrt[3]{2}$.

5. The curve $y = x(x - 1)(x - 2)$ cuts the axis at $O$, $A$, $B$, in order. Show that the areas subtended by $OA$ and $AB$ are equal.

6. Show that the area included between $Ox$ and a semi-undulation of $y = \sin x$ contains 2 units.

7. Prove that $y = \sin^2 x$ bisects the area of the rectangle formed by the axes, $x = \frac{1}{2}\pi$, $y = 1$.

8. Show that $a^n y^m = b^m x^n$ divides the rectangle formed by the axes, $x = a$, $y = b$ into two parts whose areas are in the ratio of $m : n$.

9. Find the area bounded by the curve $6xy = x^4 + 3$, the axis of $x$ and the ordinates $x = 1$, $x = 2$. *Ans.* $\frac{5}{8} + \frac{1}{2}\log 2 = 0{\cdot}972\ldots$.

10. Find the area bounded by $(1 - x^2)y = (x - 2)(x - 3)$, the axis of $x$ and the ordinates $x = 2$, $x = 3$. *Ans.* $\log(2^{11} \, . \, 3^{-6}) - 1$.

11. Find the area bounded by $a^2y(x + a) = x^4$, $x = 0$, $y = 0$, $x = a$. *Ans.* $a^2 (\log 2 - \frac{7}{12})$.

12. Find the area between $(x - a)^2y = x^2(x + a)$, $y = x + 3a$, $x = 0$, $x = -a$. *Ans.* $a^2(5 \log 2 - 1)$.

13. Show that the area between $y^2 = 4ax$ and the tangents at the ends of the latus rectum is $4a^2/3$.

14. Show that the area between $y = \sin x$, $y = \sin 2x$ and $x = \frac{1}{3}\pi$ is $\frac{1}{4}$.

15. Find the area between $y = ax/\sqrt{(x^2 + a^2)}$, $y = a$, $x = 0$, $x = a$. *Ans.* $(2 - \sqrt{2})a^2$.

16. Find the area of $5y^2 = 2a(x + a)$ cut off by the double ordinate $x = 3a/5$. *Ans.* $128a^2/75$.

17. Find the area between $y^2(a - x) = x^3$ and $x = a$. *Ans.* $3\pi a^2/4$.

18. Show that the loop of $ay^2 = x^2(a - x)$ encloses an area equal to $8a^2/15$.

19. Prove that the area enclosed by either loop of $c^2y^2 = x^2(a^2 - x^2)$ is $\frac{2}{3}a^3/c$.

20. Prove that the ordinate $x = a$ divides the area between

$$y^2(2a - x) = x^3 \quad \text{and} \quad x = 2a$$

in the ratio of $3\pi - 8 : 3\pi + 8$.

21. Show that the whole area of the curve given by

$$x = a\cos^3\theta \qquad y = a\sin^3\theta$$

is $3\pi a^2/8$.

22. Trace the curve $2x^2 - 2xy + y^2 = 4$, and show that its area is $4\pi$.

23. Show that $y = \pm x\sqrt{(1 - x^2)}$ is a curve resembling a figure of eight, and that the area of each loop is $\frac{2}{3}$.

24. Show that $x = 4a$ cuts off from the parabola $(2x - y)^2 = 4ax$ a segment whose area is $64a^2/3$.

## EXERCISES X (B)

### Volumes

1. Show that a sphere of radius 1 foot is divided by a plane distant 4 in. from the centre into two parts whose volumes are in the ratio of $7 : 20$.

2. Prove that the volume of the segment of a sphere whose height is $h$ and the radius of whose base is $c$ is $\frac{1}{6}\pi h(3c^2 + h^2)$.

3. Find the volume of the spheroid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis. *Ans.* $64\pi/3$.

4. The segment of the parabola $y^2 = 4ax$ lying between $x = a$, $x = 2a$ is revolved about $Ox$. Show that the volume generated is $6\pi a^3$.

5. Find the volume generated by the revolution about $Ox$ of the areas defined in Question 1, Exercises X (A). *Ans.* $\frac{1}{5}\pi$, $\frac{1}{4}\pi$.

6. Prove that the volumes generated by the revolution about $Ox$ of the loops of the two curves

$$y^2 = x(x - 1)(x - 2) \qquad y^2 = x(1 - x)(x - 2)$$

are equal.

7. The area bounded by $a^2y = x^2(x - a)$, the axes and $x = a$ is revolved about $Ox$. Show that the volume generated is $\pi a^3/105$.

8. The area bounded by $y^2x = 4a^2(2a - x)$ and the ordinates $x = a$, $x = 2a$ is revolved about $Ox$. Show that the volume generated is $4\pi a^3(2 \log 2 - 1)$.

9. Show that the volume generated by the revolution about the latus rectum of the segment of the parabola (latus rectum $= 4a$) cut off by its latus rectum is $32\pi a^3/15$.

10. The area bounded by $x^2 = y^2(2 - y)$, the axes and $y = 2$ is revolved about $Oy$. Show that the volume generated is $4\pi/3$.

11. Prove that the volume generated by the revolution of

$$y^2(a^2 + x^2)^2 = 2a^5x$$

about $Ox$ is $\pi a^3$.

12. Show that the volume generated by the revolution of

$$x^2/a^2 + y^2/b^2 = 1$$

about the tangent at the end of the major axis is $2\pi^2a^2b$.

13. A cylindrical hole of length $2c$ is bored centrically through a sphere. Prove that the volume of the part left is $4\pi c^3/3$.

14. Prove that the volume generated by revolving about its axis the segment of a parabola cut off by the latus rectum bears to the volume generated by revolving the same area about the latus rectum the ratio of 15 : 16.

## EXERCISES X (C)

### Simpson's Rules

1. If Simpson's rule is applied to the calculation of the areas in Question 1, Exercises X(A), show that in the first case the correct answer is found, and that in the second case the error is about 0·5 per cent.

2. If Simpson's rule is applied to the calculation of the volumes in Question 1, Exercises X (B), show that the correct answer is given.

3. Apply Simpson's rule to determine the area between $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{1}{3}\pi$. Show that the answer is in excess by about $4 \times 10^{-4}$.

4. If the trapezoidal area defined by the curve

$$y = \frac{x + 2}{(x + 1)(4 - x)}$$

the axis of $x$, and (i) $x = -4$, $x = -2$; (ii) $x = 0$, $x = 3$; (iii) $x = 5$, $x = 7$ be denoted with positive signs by $A_1$, $A_2$, $A_3$ respectively, prove by integration that the values of $A_1$, $A_2$, $A_3$ are 0·125..., 1·94..., 1·261— respectively, and by Simpson's approximative rule are 0·123..., 1·99..., 1·276—.

5. By applying Simpson's rule to calculate the area bounded by $xy = 1$, $y = 0$, $x = 1$, $x = 2$, find an approximate value of log 2.

6. Show that Simpson's rule is correct when applied to a curve $y = f(x)$, when $f(x) = 1, x, x^2, x^3$, and also when

$$f(x) = a + bx + cx^2 + dx^3$$

7. If $y_1, y_2, y_3, y_4$ are four equidistant positive ordinates of the curve

$$y = a + bx + cx^2 + dx^3$$

given by $x = 0, h, 2h, 3h$, show that the area between the curve, the axes and $x = 3h$ is

$$\tfrac{3}{8}(y_1 + 3y_2 + 3y_3 + y_4)h$$

8. If $y_1, y_2, y_3$ are ordinates of a curve which lies above $Ox$, and if the distances between the consecutive pairs of ordinates are $a, b$, prove that the area bounded by the curve, $Ox$ and the extreme ordinates is approximately

$$\frac{a+b}{6ab}\{(2a - b)by_1 + (a + b)^2y_2 + (2b - a)ay_3\}$$

Find the corresponding formula for the volume generated by the revolution of this area about $Ox$.

9. If $r_1, r_2, r_3$ are the three radii of the curve

$$r^2 = a + b\theta + c\theta^2 + d\theta^3$$

defined by the angles $\beta$, $\beta + \alpha$, $\beta + 2\alpha$, show that the sectorial area bounded by $r_1, r_3$ and the curve is

$$\tfrac{1}{6}(r_1^2 + 4r_2^2 + r_3^2)\alpha$$

10. Given that $y_1, y_2, y_3$ are the radii of equidistant cross-sections of a surface of revolution, prove that the moment of inertia about its axis of the volume bounded by the first and last section is approximately

$$\tfrac{1}{6}\pi(y_1^4 + 4y_2^4 + y_3^4)h$$

# CHAPTER XI

## MOMENTS BY INTEGRATION

**122. The moments of a system of particles distributed along Ox.**

Let $m_1$ be at $x_1$, $m_2$ at $x_2$, ..., then the *first moment* of the system about $O$ is
$$m_1x_1 + m_2x_2 + \dots = \Sigma mx = \mathrm{M}_1$$

Again, the *second moment, quadratic moment* or *moment of inertia* about $O$ is
$$m_1x_1^2 + m_2x_2^2 + \dots = \Sigma mx^2 = \mathrm{M}_2$$
The mass of the system is $\Sigma m$.

**123. First and second moments of a line-distribution of matter.**

Let the matter be distributed along $AB$, where $x = a$, $x = b$ give the coordinates of the terminal points of the distribution, and let the density at $P$ be $\rho(x)$ or $\rho$, as we may sometimes call it.

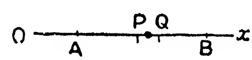


FIG. 58.

Now, if $\rho$ is constant, the mass of the element $PQ$ ($= \delta x$) is $\rho\, \delta x$, but if $\rho$ varies and is continuous there is some point $x + \theta\, \delta x$ in $PQ$ at which the density $\rho(x + \theta\, \delta x)$, when multiplied by $\delta x$, gives the mass of $PQ$.* Hence the total mass is $\Sigma \rho(x + \theta\, \delta x)\delta x$.

Again, if we take the line $AB$ and cut it up into lengths equal to $\delta x$, we can find points in each small section at which the mass of the element must be placed to give a system of a finite number of masses which has the same first moment about $O$ as the given continuous line-distribution. Let the point of $PQ$ at which $\rho(x + \theta\, \delta x)\delta x$ is placed be $x + \theta'\delta x$, where $\theta$, $\theta'$ are both positive proper fractions. The first moment of $PQ$ about $O$ is now
$$\rho(x + \theta\, \delta x) \,.\, (x + \theta'\delta x)\delta x$$

Also, if $\mathrm{M}_1(x)$ is the first moment of the matter along $AP$ about $O$, then $\mathrm{M}_1(x + \delta x)$ is the first moment of $AQ$, and
$$\begin{aligned}\delta \mathrm{M}_1(x) &= \text{first moment of } PQ\\ &= \rho(x + \theta\, \delta x) \,.\, (x + \theta'\delta x)\delta x\end{aligned}$$
whence
$$\frac{\delta \mathrm{M}_1(x)}{\delta x} = \rho(x + \theta\, \delta x) \,.\, (x + \theta'\delta x)$$

* The reader may compare the argument used in Art. 108.

Proceeding to the limit in which $\delta x \to 0$, we have

$$\frac{dM_1}{dx} = \rho(x) \,.\, x$$

Again, if we take $M_2(x)$ as the second moment of $AP$ about $Ox$, we find by a similar argument that

$$\delta M_2(x) = \rho(x + \theta\,\delta x) \,.\, (x + \theta''\delta x)^2 \delta x$$

where $\theta$ is the same as before, but $\theta''$ is another proper fraction which is not in general the same as $\theta'$. Whence we obtain

$$\frac{dM_2}{dx} = \rho(x) \,.\, x^2$$

If $m(x)$ is the mass of $AP$, we also have

$$\frac{dm}{dx} = \rho(x)$$

These three differential coefficients of $M_1$, $M_2$, $m$ allow us, by integration and determination of the constants which occur, to find the first and second moments and the mass of $AP$, the constants being determined by

$$M_1(a) = 0 \qquad M_2(a) = 0 \qquad m(a) = 0$$

The first moment of $AB$ is now $M_1(b)$, its second moment $M_2(b)$ and its mass $m(b)$.

**124. Illustrative examples.**

Ex. 1. *To find the first and second moments of a rod of uniform section about a point $O$, distant $a$ from one end, $O$ lying in the line of the rod produced.*

Let $OA = a$, $OB = b$ give the ends ; $\rho$ being constant. We have

$$\frac{dM_1}{dx} = \rho x$$

$$M_1(x) = \tfrac{1}{2}\rho x^2 + C$$

and $M_1(a) = 0$ ; therefore

$$M_1(x) = \tfrac{1}{2}\rho(x^2 - a^2)$$

and $$M_1(b) = \tfrac{1}{2}\rho(b^2 - a^2) = \tfrac{1}{2}m(a + b)$$

when $m$ is the mass of the rod.

Again, for the second moment we have

$$\frac{dM_2}{dx} = \rho x^2$$

$$M_2(x) = \tfrac{1}{3}\rho x^3 + C$$

$$= \tfrac{1}{3}\rho(x^3 - a^3)$$

$$M_2(b) = \tfrac{1}{3}m(b^2 + ba + a^2)$$

Ex. 2. *To find the first and second moments about one end of a rod whose density varies as the square of the distance from that end.*

This is the case of a conical rod tapering to an end, which is taken as origin ; $\rho = kx^2$.

$$\frac{dM_1}{dx} = \rho x = kx^3$$

$$M_1(x) = \tfrac{1}{4}kx^4, \text{ since } M_1(0) = 0$$

$$M_1(l) = \tfrac{1}{4}kl^4$$

Again, if $m(x)$ is the mass of a length $x$,

$$\frac{dm}{dx} = kx^2$$

$$m(x) = \tfrac{1}{3}kx^3$$

and $m$, the mass of the rod, $= \tfrac{1}{3}kl^3$

whence $M_1(l) = \tfrac{3}{4}ml$

Again, $\dfrac{dM_2}{dx} = \rho x^2 = kx^4$

$$M_2(x) = \tfrac{1}{5}kx^5$$

$$M_2(l) = \tfrac{1}{5}kl^5 = \tfrac{3}{5}ml^2$$

Ex. 3. *To find the second moment of a tapering conical rod about the thick end.*

Here we take $\rho = kx^2$ as in Ex. 2, and to get the second moment of the element at $P$ we multiply by $(l - x)^2$. Thus

$$\frac{dM_2}{dx} = kx^2(l - x)^2 = k(l^2x^2 - 2lx^3 + x^4)$$

$$M_2(x) = k(\tfrac{1}{3}l^2x^3 - \tfrac{1}{2}lx^4 + \tfrac{1}{5}x^5), \text{ since } M_2(0) = 0$$

$$M_2(l) = \tfrac{1}{30}kl^5 = \tfrac{1}{10}ml^2$$

**125. First and second moments of a lamina.**

The discussion of the moments of a plane distribution of matter about an axis $Ox$ (or $Oy$) will now be sketched. In its general form the problem is a double summation. The area is divided into a large number of small rectangles whose sides are parallel to the axes and equal to $\delta x$ and $\delta y$ in length. If $\sigma$ is the density at $(x, y)$, the mass of the elementary rectangle, in which $(x, y)$ is situated, is taken as $\sigma\,\delta x\,\delta y$ ; its first moment about $Oy$ is $\sigma x\,\delta x\,\delta y$. The first moment of the whole lamina about $Oy$ is the limit of the sum of all such elements as $\sigma x\,\delta x\,\delta y$, when the number of the rectangles is increased indefinitely, their length and breadth being decreased indefinitely. Under the same circumstances, the mass of the

lamina is the limit of the sum of the terms $\sigma\,\delta x\,\delta y$. Also its second moment about $Oy$ is the limit of the sum of $\sigma x^2 \delta x\,\delta y$.

We may simplify the discussion by considering an area lying upon one side of $Ox$ which is bounded by $y = f(x)$, $Ox$ and a couple of ordinates and by taking the case in which $\sigma$ is either constant or a function of $x$ only.

Thus, taking $M_1(x)$ as the first moment of $AA'P'P$ about $Oy$, we have that $M_1(x + \delta x)$ is the first moment of $AA'Q'Q$ about $Oy$; therefore

$$\begin{aligned}\delta M_1(x) &= M_1(x + \delta x) - M_1(x)\\ &= \text{first moment about } Oy \text{ of } PP'Q'Q\\ &= \sigma y\,\delta x\,.\,x, \text{ approximately}\end{aligned}$$

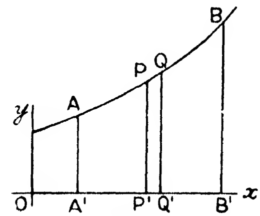


FIG. 59.

and $y = f(x)$. Whence

$$\frac{dM_1}{dx} = \sigma(x)\,.\,xy$$

Again, taking $M_2(x)$ as the second moment, we deduce that

$$\frac{dM_2}{dx} = \sigma(x)\,.\,x^2 y$$

The above method applies to the case of the standard figure given in Fig. 59. But for such a figure the process of determining moments about $Ox$ would not be so simple, even in the case of $\sigma$ constant. The areas to which we shall usually apply our method are of the nature of the ellipse in which the perpendiculars from $P$ upon $Ox$, $Oy$ both lie in the area. For such areas we shall introduce the symbols $N_1(y)$, $N_2(y)$ to denote the first and second moments about $Ox$ of areas lying between two parallels to $Ox$, the nearer of which is fixed and the further is given by $y$. In this case we have, if $\sigma$ is constant or a function of $y$, that

$$\frac{dN_1}{dy} = \sigma\,xy \qquad \frac{dN_2}{dy} = \sigma x y^2$$

where $x = \chi(y)$ is the equation of the bounding curve.

The four formulae

$$\frac{dM_1}{dx} = \sigma x f(x) \qquad \frac{dM_2}{dx} = \sigma x^2 f(x)$$

$$\frac{dN_1}{dy} = \sigma y \chi(y) \qquad \frac{dN_2}{dy} = \sigma y^2 \chi(y)$$

(where $y = f(x)$, $x = \chi(y)$ are the same functional relation), apply only when $\sigma$ is constant. We may add that in the case of constant density, $m$ (the total mass) $= \sigma$ . area.

**126. Illustrative examples.**

Ex. 1. *The first and second moments of a uniform rectangular lamina about adjacent sides.*

Let $a$, $b$ be the length and breadth and let $\sigma = 1$, then

$$\frac{dM_1}{dx} = bx$$

$$M_1(x) = \tfrac{1}{2}bx^2, \text{ since } M_1(0) = 0$$

Thus $$M_1(a) = \tfrac{1}{2}a^2b = \tfrac{1}{2}ma$$

Similarly $$N_1(b) = \tfrac{1}{2}mb$$

Again $$\frac{dN_2}{dy} = ay^2$$

$$N_2(y) = \tfrac{1}{3}ay^3$$

$$N_2(b) = \tfrac{1}{3}ab^3 = \tfrac{1}{3}mb^2$$

Similarly $$M_2(a) = \tfrac{1}{3}ma^2$$

Ex. 2. *The first moment about a bounding radius of the quadrant of a circle.*

As before, we take $\sigma = 1$, and the equation of the circle as

$$y = \sqrt{(a^2 - x^2)}$$

Hence $$\frac{dM_1}{dx} = xy = x\sqrt{(a^2 - x^2)}$$

$$M_1(x) = C - \tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}}$$

Now $$M_1(0) = 0, \text{ therefore } C = \tfrac{1}{3}a^3$$

$$M_1(x) = \tfrac{1}{3}a^3 - \tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}}$$

and $$M_1(a) = \tfrac{1}{3}a^3 = 4ma/3\pi$$

The equation to which the expression for the second moment leads cannot be integrated without the use of circular functions. The equation is

$$\frac{dM_2}{dx} = x^2y = x^2\sqrt{(a^2 - x^2)}$$

The solution (which is given in Chap. XIII) is

$$M_2(x) = \tfrac{1}{8}a^4\sin^{-1}(x/a) + \tfrac{1}{8}x(2x^2 - a^2)\sqrt{(a^2 - x^2)}$$

From this $$M_2(a) = \tfrac{1}{16}\pi a^4 = \tfrac{1}{4}ma^2$$

**127. Second, or axial, moment of a lamina about an axis perpendicular to its plane.**

If the axes of reference of the lamina intersect at the point at which the perpendicular axis meets it, then the required second moment is the $\lim \Sigma\sigma r^2\delta x\,\delta y = \lim \Sigma\sigma x^2\delta x\,\delta y + \lim \Sigma\sigma y^2\delta x\,\delta y$ (for $r^2 = x^2 + y^2$), and this is the sum of the second moments of the lamina about two axes intersecting at right angles in the point at which the axis perpendicular to the lamina meets its plane.

It is an important deduction from the result given in Art. 126, Ex. 2, that the second moment of a uniform circular lamina about its axis is

$$M_2(a) + N_2(a) = \tfrac{1}{4}ma^2 + \tfrac{1}{4}ma^2 = \tfrac{1}{2}ma^2$$

This result is of such importance that it may be proved independently of integrations which have not yet been performed in the following way. Let $I(r)$ be the second moment of a circular lamina of radius $r$ about its axis, then $I(r + \delta r)$ is that of a circle of radius $(r + \delta r)$; therefore $I(r + \delta r) - I(r) = \delta I(r)$ is the second moment of an annulus bounded by circles of radii $r$, $r + \delta r$. But this second moment is equal to the mass of the annulus multiplied by the square of a mean radius.

$$\delta I(r) = 2\pi\sigma(r + \theta_1 \delta r)\,\delta r(r + \theta\,\delta r)^2$$

that is
$$\frac{dI}{dr} = 2\pi\sigma r^3$$

$$I = \tfrac{1}{2}\pi\sigma r^4, \text{ since } I(0) = 0$$

$$I(a) = \tfrac{1}{2}\pi\sigma a^4 = \tfrac{1}{2}ma^2$$

It follows from Art. 126, Ex. 1, and from the statement at the beginning of the article, that the second moment of a uniform rectangular lamina whose sides are $2a$, $2b$ about an axis through its centre perpendicular to its plane is

$$\tfrac{4}{3}ab(a^2 + b^2)\sigma = \tfrac{1}{3}m(a^2 + b^2)$$

also, the second moment of a square lamina (edge $= 2a$) about its central axis is

$$\tfrac{2}{3}ma^2$$

**128. Second, or axial, moments of volumes about an axis of symmetry.**

First, we take volumes bounded by surfaces of revolution and write $I(x)$ for the second moment about $Ox$ of the volume generated by $A'P'PA$ (see Fig. 59). Then we have

$$\delta I(x) = \text{second moment of the volume generated by the revolution of } PP'Q'Q$$

$$= \tfrac{1}{2}\pi y^4 \delta x$$

$$\frac{dI}{dx} = \tfrac{1}{2}\pi y^4$$

By integration $I(x)$ is found, $I(a)$ being zero, and if the end-planes are given by $x = a$, $x = b$, the required second moment is $I(b)$.

As a second problem, we take a cuboid whose edges are $2a$, $2b$, $2c$, and find its second moment about the $c$-axis. In this case, by the expression in Art. 127 for the second moment of a rectangular lamina, we deduce that

$$\delta I(x) = \tfrac{4}{3}ab(a^2 + b^2)\,\delta x$$

$$I(x) = \tfrac{4}{3}abx(a^2 + b^2)$$

if $x$ is measured from a face

$$\begin{aligned} \mathrm{I}(2c) &= \tfrac{8}{3}abc(a^2 + b^2) \\ &= \tfrac{1}{3}(a^2 + b^2) \times \text{volume} \end{aligned}$$

**129. Illustrative examples.**

Ex. 1. *To find the second moment of a cone about its axis.*

We have
$$\begin{aligned} \frac{d\mathrm{I}}{dx} &= \tfrac{1}{2}\pi y^4 = \tfrac{1}{2}\pi x^4 \tan^4 \alpha \\ \mathrm{I}(x) &= \tfrac{1}{10}\pi x^5 \tan^4 \alpha \\ \mathrm{I}(h) &= \tfrac{1}{10}\pi h^5 \tan^4 \alpha = \tfrac{3}{10} r^2 \mathrm{V} \end{aligned}$$

Ex. 2. *To find the second moment of a hemisphere about its axis.*

Here
$$\begin{aligned} \frac{d\mathrm{I}}{dx} &= \tfrac{1}{2}\pi y^4 = \tfrac{1}{2}\pi(a^2 - x^2)^2 \\ &= \tfrac{1}{2}\pi(a^4 - 2a^2x^2 + x^4) \\ \mathrm{I}(x) &= \tfrac{1}{2}\pi(a^4 x - \tfrac{2}{3}a^2x^3 + \tfrac{1}{5}x^5), \text{ since } \mathrm{I}(0) = 0 \\ \mathrm{I}(a) &= 4\pi a^5/15 = \tfrac{2}{5}a^2\mathrm{V} \end{aligned}$$

Ex. 3. *To find the second moment of a pyramid on a square base about the axis through the vertex perpendicular to the base.*

Let $2y$ be the side of a square section at a distance $x$ from the vertex then
$$\begin{aligned} \frac{d\mathrm{I}}{dx} &= \tfrac{8}{3}y^4 = \tfrac{8}{3}a^4x^4/h^4 \\ \mathrm{I}(x) &= \tfrac{8}{15}a^4x^5/h^4 \\ \mathrm{I}(h) &= \tfrac{8}{15}a^4h = \tfrac{2}{5}a^2\mathrm{V} \end{aligned}$$

**130. Centroid or centre of gravity.**

In the case of a mass-distribution certain quantities called first moments have been investigated; if we take the case of the lamina and write
$$\mathrm{M}_1(a) = m\mathrm{X} \qquad \mathrm{N}_1(b) = mY$$
we have a point (X, Y) which is called the mass-centre or centre of gravity of the distribution. If we take a uniform surface distribution, we may disregard the notion of mass, as (X, Y) is then determined by the shape of the area only; the point is then called the centroid of the area.

A point with corresponding geometrical and physical properties exists for every solid and every solid mass, but at present we shall restrict ourselves to the case of a solid bounded by a surface of revolution; we take a plane of reference perpendicular to the axis of revolution and construct a sum by multiplying all the small elements of the volume by their distances ($x$) from the plane of reference. Thus we obtain the conception of a first moment of a volume with regard to a plane. If a section is made by a plane whose coordinate is $x$, such that the volume included by it and the

plane of reference is $V(x)$, and if we call the first moment of this volume with regard to the plane of reference $M_1(x)$, then we have by the method followed throughout this chapter

$$\delta M_1(x) = x \times \text{volume of a slice of the solid bounded by the planes whose coordinates are } x \text{ and } x + \delta x$$
$$= \pi x y^2 \delta x$$

Therefore $$\frac{dM_1}{dx} = \pi x y^2 = \pi x [f(x)]^2$$

where $y = f(x)$ is the equation of the generating curve.

Now, if the volume is bounded by the planes $x = a$, $x = b$, its first moment is

$$M_1(b)$$

and the $x$-coordinate of the centroid is

$$M_1(b)/V$$

**131. Illustrative example.**

Ex. *To find the centroid of a hemisphere.*

Now $$\frac{dM_1}{dx} = \pi x y^2 = \pi(a^2x - x^3)$$
$$M_1(x) = \pi(\tfrac{1}{2}a^2x^2 - \tfrac{1}{4}x^4), \text{ since } M_1(0) = 0$$
$$M_1(a) = \tfrac{1}{4}\pi a^4$$

and $$X = M_1(a)/V = \tfrac{1}{4}\pi a^4/\tfrac{2}{3}\pi a^3 = \tfrac{3}{8}a$$

## EXERCISES XI

1. Taking $m$ as the mass of a rod $OA$, whose length is $l$ and whose mass-centre is at $G$, prove that when $\rho = kx$

i. with $O$ as base-point, $M_1 = \frac{2}{3}ml$, $M_2 = \frac{1}{2}ml^2$
ii. with $A$ as base-point, $M_1 = \frac{1}{3}ml$, $M_2 = \frac{1}{6}ml^2$
iii. with $G$ as base-point, $M_1 = 0$, $M_2 = \frac{1}{18}ml^2$

2. When $\rho = kx^2$, prove that

i. with $O$ as base-point, $M_1 = \frac{3}{4}ml$, $M_2 = \frac{3}{5}ml^2$
ii. with $A$ as base-point, $M_1 = \frac{1}{4}ml$, $M_2 = \frac{1}{10}ml^2$
iii. with $G$ as base-point, $M_1 = 0$, $M_2 = \frac{3}{80}ml^2$

3. The thickness at $(x, y)$, a point of a rectangular lamina whose sides are $x = \pm a$, $y = \pm b$, is $t(1 - x^2/a^2)$. Show that $m = \frac{8}{3}tab$, $M_2 = \frac{1}{5}ma^2$.

4. The shape of a uniform lamina is one-half of a parabolic segment cut off from $y^2 = 4ax$ by $x = x_1$. Show that

$$M_1 = \tfrac{3}{5}mx_1 \qquad N_1 = \tfrac{3}{8}my_1 \qquad M_2 = \tfrac{3}{7}mx_1^2 \qquad N_2 = \tfrac{1}{5}my_1^2$$

5. A segment of a parabola $y^2 = 4ax$, whose furthest corner is $(x_1, y_1)$, revolves about $Ox$. Show that the moment of inertia about the axis of the uniform solid paraboloid thus defined is $\frac{1}{3}my_1^2$.

# ANSWERS

## Exercises I. P. 16

**2.** i. $-\frac{1}{2}x+1$ ii. $-3x+4$ iii. $0x-3$ iv. $2x$ v. $x$
vi. $0x+2$ vii. $-x+7$ viii. $x+6$ ix. $-5x$

**4.** i. $\frac{1}{4}x^2$ ii. $x^2+5$ iii. $-x^2+x+1$ iv. $x^2-x+1$
v. $x^2+2x$ vi. $13x^2-45x+30$ vii. $0{\cdot}9x^2+2x-2$ viii. $4x^2-6x+2$

**5.** i. $[-\infty, -1)$ ii. $[-\infty, -2{\cdot}5]$ iii. $[1, \infty]$ iv. $[-\infty, 1)$
v. $[-\infty, 1]$ vi. $[3, \infty]$ vii. $[-\infty, 1][3, \infty]$
viii. $(-1, -2)$ ix. $(0, 2{\cdot}5)$ x. $[0, 2{\cdot}5]$ xi. $[-1, 1{\cdot}5]$
xii. $[-\infty, -2\frac{1}{2})(\frac{2}{3}, \infty]$ xiii. $[-2\frac{1}{2}, \frac{2}{3}]$ xiv. $(-2\frac{1}{2}, 0](\frac{2}{3}, \infty]$
xv. $[-\infty, -1][0, 1]$ xvi. $[-1, 0][\frac{1}{3}, \infty]$ xvii. $[-\infty, 0][0, 2]$
xviii $[0, 2]$ xix. $(-2, 7]$ xx. $(-1, 3]$ xxi. $(0, 4]$
xxii. $(0, 1]$ xxiii. $[-\infty, -10][-2, 2][4, \infty]$ xxiv. $[-\infty, -3](-2, 0](1, \infty]$
xxv. $[-2, 1][1, 2]$ xxvi. $[-2, -1][-1, -\frac{1}{12}]$

**7.** $\frac{1}{2}$; $\frac{1}{3}, \frac{2}{3}$; $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$; $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$; ..

## Exercises II. P. 33

**2.** i. $x=-1$, L lim $=-\infty$, R lim $=\infty$ ii. $x=-1$, no L lim, R lim $=\infty$
iii. $x=-1$, L lim $=\infty$, R lim $=-\infty$ iv. $x=-1$, L lim $=$ R lim $=0$
v. $x=-1$, no L lim, R lim $=-\infty$; $x=1$, L lim $=\infty$, no R lim
vi. $x=-1$, L lim $=\infty$, R lim $=\infty$; $x=0$, L lim $=$ R lim $=1$
vii. $x=-1$, L lim $=-\infty$, no R lim; $x=1$, no L lim, R lim $=0$
viii. $x=-4$, L lim $=-\infty$, R lim $=\infty$; $x=1$, L lim $=$ R lim $=0$
ix. $x=2n\pi$; L lim $=-\infty$, R lim $=\infty$; $x=(2n+1)\pi$, L lim $=\infty$, R lim $=-\infty$
x. $x=\frac{1}{2}(2n+1)\pi$, L lim $=\infty$, R lim $=\infty$
xi. Defined in $[n\pi, \frac{1}{2}(2n+1)\pi]$; $x=n\pi$, R lim $=-\infty$; $x=\frac{1}{2}(2n+1)\pi$, L lim $=\infty$
xii. $x=n\pi$, L lim $=-\infty$, R lim $=-\infty$; $x=\frac{1}{2}(2n+1)\pi$, L lim $=\infty$, R lim $=\infty$

**3.** i. $(-1, \infty]$ ii. $[-\infty, -1)(1, \infty]$ iii. $(-2, 2)$ iv. $[-\infty, 1)(2, \infty$
v. $[-\infty, 0](1, \infty]$ vi. $[-\infty, 0][0, 1][1, \infty]$ vii. $(-1, 0](1, \infty]$
viii. .. $(0, \pi)(2\pi, 3\pi)$ ... ix. ... $(-\frac{7}{6}\pi, \frac{1}{6}\pi)(\frac{5}{6}\pi, \frac{13}{6}\pi)$ ... x. $[0, \infty]$
xi. $(1, \infty]$ xii. ... $[0, \frac{1}{2}\pi][\pi, \frac{3}{2}\pi]$ ... xiii. ... $[0, \frac{1}{2}\pi][\frac{1}{2}\pi, \pi]$ ...
xiv. $[0, 1][1, \infty]$ xv. ... $[\frac{1}{4}\pi, \frac{5}{4}\pi][\frac{9}{4}\pi, \frac{13}{4}\pi]$ . xvi. $[1, \infty]$

**6.** i. $0$ ii. $1$ iii. $1$ iv. $2{\cdot}718$ ... v. $0, 2$ vi. $\pm 1$

## Exercises III (A). P. 37

**1.** $2$ **2.** $2x$ **3.** $6x+1$ **4.** $-2x^{-3}$; $[-\infty, 0][0, \infty]$
**5.** $-(x+1)^{-2}$; $[-\infty, -1][-1, \infty]$ **6.** $5x^4$ **7.** $8x+4$

**8.** $-2x(x^2-1)^{-2}$; $[-\infty, -1][-1, 1][1, \infty]$ **9.** $\frac{1}{2}(x+1)^{-\frac{1}{2}}$; $[-1, \infty]$

**10.** $-(3-2x)^{-\frac{1}{2}}$; $[-\infty, 1\frac{1}{2}]$ **11.** $-4x(x^2+1)^{-3}$ **12.** $-(2x+1)^{-\frac{3}{2}}$; $[-\frac{1}{2}, \infty]$

## Exercises III (B). P. 46

i. $-\dfrac{2a}{(a+x)^2}$ ii. $\dfrac{4a^2x}{(a^2+x^2)^2}$ iii. $\dfrac{2a}{(a-x)^2}$ iv. $-\dfrac{4a^2x}{(x^2-a^2)^2}$

v. $\dfrac{2(1-x^2)}{(1-x+x^2)^2}$ vi. $\dfrac{2(x^2-1)}{(1+x+x^2)^2}$ vii. $\dfrac{4}{(1-x)^2}$ viii. $-\dfrac{3}{(1+x)^2}$

ix. $\dfrac{2(1+x)}{(1-x)^3}$ x. $\dfrac{2(x^2-1)}{(1+x^2)^2}$ xi. $1-\dfrac{2}{(x+3)^2}$ xii. $\dfrac{-(x^2+6x+7)}{(x+1)^2(x+2)^2}$

xiii. $\dfrac{-(x-a)^2-(x-b)^2}{(x-a)^2(x-b)^2}$ xiv. $\dfrac{4(1-2x)}{(x-3)^2(x+2)^2}$ xv. $\dfrac{4(2x-1)}{(x+1)^2(x-2)^2}$ xvi. $n\dfrac{x^{2n}+a^{2n}}{x^{n+1}}$

xvii. $-1$ xviii. $\dfrac{1}{\sqrt{x}(1-\sqrt{x})^2}$ xix. $-\dfrac{1}{\sqrt{x}(1+\sqrt{x})^2}$ xx. $\sin x(2-3\sin^2 x)$

xxi. $\cos x - x\sin x$ xxii. $\dfrac{\sin x - x\cos x}{\sin^2 x}$ xxiii. $\dfrac{2\cos x}{(1-\sin x)^2}$ xxiv. $\dfrac{1+\sin x}{\cos^2 x}$

xxv. $\dfrac{\sin^3 x \quad \cos^3 x}{\sin^2 x\cos^2 x}$ xxvi. $\dfrac{2}{(\sin x+\cos x)^2}$ xxvii. $\dfrac{2}{(\sin x-\cos x)^2}$

## Exercises IV. P. 56

**1.** i. $x=\frac{1}{3}$, min. ii. $x=\frac{1}{2}$, max.; $x=2\frac{1}{2}$, min.
iii. $x=1$, min.; $x=5$, max. iv. $x=3$, max.; $x=5$, min.
v. $x=-4$, min. vi. $x=0{\cdot}31$, $3{\cdot}19$, max.; $x=1$, min.
vii. $x=-2$, $1$, min.; $x=-1$, max. viii. $x=1$, max.; $x=3$, min.

**2.** i. $x=-3$, max.; $x=0$, min. ii. $x=-2$, min. iii. $x=1$, min.

**3.** i. $f(2)=0$, min.; $f(2\frac{1}{3})=1/27$, max. ii. $f(0)=0$, $f(2)=0$, min.; $f(1)=1$, max.
iii. $f(0)=0$, stationary; $f(-\frac{3}{4})=-0{\cdot}105$, min.
iv. $f(-1)=0$, stationary; $f(0{\cdot}2)=1{\cdot}106$, max.; $f(1)=0$, min.

**5.** $x^3+6x^2-15x$

**6.** i. $f(1)=f(-1)=2$, min. ii. $f(\frac{1}{4})=0{\cdot}844$, min.
iii. $f(-1)=3$, max.; $f(1)=\frac{1}{3}$, min.
iv. $f(\frac{1}{2}(-1\pm\sqrt{5}))=-\frac{1}{2}$, min.; $f(\frac{1}{2}(1\pm\sqrt{5}))=\frac{1}{2}$, max.

**7.** i. $x=2$, min.; $x=6$, max. ii. $x=\frac{1}{2}(-\sqrt{2}\pm\sqrt{6})$, min.; $x=\frac{1}{2}(\sqrt{2}\pm\sqrt{6})$, max.
iii. $x=-\sqrt{5}$, max. $(-\frac{1}{2}5\sqrt{5})$; $x=\sqrt{2}$, max. $(4\sqrt{2})$; $x=-\sqrt{2}$, min. $(-4\sqrt{2})$; $x=\sqrt{5}$, min. $(\frac{1}{2}5\sqrt{5})$

**8.** i. $x=(2n+\frac{1}{4})\pi$, max.; $x=(2n+\frac{5}{4})\pi$, min.
ii. $x=(2n+\frac{1}{3})\pi$, max.; $x=(2n-\frac{1}{3})\pi$, min.
iii. $x=\frac{1}{2}(4n-1)\pi$, max.; $x=\frac{1}{2}(4n+1)\pi$, min.
iv. $x=\frac{1}{2}(4n-1)\pi$, $n\pi+(-1)^n\sin^{-1}(1/\sqrt{6})$, max.
$x=\frac{1}{2}(4n+1)\pi$, $n\pi-(-1)^n\sin^{-1}(1/\sqrt{6})$, min.
v. $x=(2n+1)\pi$, $2n\pi\pm\cos^{-1}(1/\sqrt{3})$, max.
$x=2n\pi$, $(2n+1)\pi\pm\cos^{-1}(1/\sqrt{3})$, min.
vi. $x=n\pi+\frac{1}{2}a$, max.; $x=(n+\frac{1}{2})\pi+\frac{1}{2}a$, min.

## Exercises V. P. 72

**1.** i. $3(x-1)^2-2$  ii. $2(x-1)^2+32$  iii. $7(x+1)^2-14$
iv. $4-(x+3)^2$  v. $(x+1)^2+2$  vi. $(x-1)^2-4$

**4.** i. $(x^2+x+1)(x^2-x+1)$
ii. $(x^2-x-1)(x^2+x-1)$, $(x^2-\sqrt{5}x+1)(x^2+\sqrt{5}x+1)$
iii. $(x^2+\sqrt{6}x+3)(x^2-\sqrt{6}x+3)$

**7.** i. $3\frac{7}{9}$  ii. $-4\frac{1}{9}$  iii. $4, -\frac{2}{3}, -6\frac{5}{12}$

**9.** i. $R(-2)=1\frac{1}{3}$, max.; $R(0)=0$, min.  ii. $R(0)=\frac{1}{4}$, max.
iii. $R(-6)=1\frac{1}{3}$, max.; $R(1\frac{1}{2})=-\frac{1}{3}$, min.
iv. $R(1\frac{1}{2})=-3$, max.; $R(-6)=\frac{3}{4}$, min.  v. $R(-4)=0$, min.
vi. $R(-8)=-14$, max.; $R(-2)=-2$, min.
vii. $R(-3)=1\frac{1}{2}$, max.; $R(3)=\frac{3}{4}$, min.
viii. $R(4{\cdot}4)=-31{\cdot}5$, max.; $R(2{\cdot}18)=-0{\cdot}5$, min.
ix. None  x. $R(-1)=1\frac{1}{3}$, max.
xi. $R(-b/a)=0$, max., $R(b/a)=4ab$, min., if $b/a$ is positive
xii. $R(-2+\sqrt{3})=2+\sqrt{3}$, max.; $R(-2-\sqrt{3})=2-\sqrt{3}$, min.

## Exercises VI (B). P. 82

**1.** $4(2x+1)$  **2.** $4(2x-1)$  **3.** $-9(2-3x)^2$  **4.** $-\frac{9}{2}\sqrt{(2-3x)}$

**5.** $\dfrac{2(x^4-1)}{x^3}$  **6.** $\dfrac{x^2+1}{2x^{\frac{3}{2}}\sqrt{(x^2-1)}}$  **7.** $\dfrac{6x(1+2x)^2}{(1+3x)^3}$

**8.** $\dfrac{-6x(1-3x)}{(1-2x)^4}$  **9.** $\dfrac{-1}{(1+x)\sqrt{(1-x^2)}}$  **10.** $\dfrac{2x}{(1-x^2)\sqrt{(1-x^4)}}$

**11.** $\dfrac{a^2}{\sqrt{(a^2-x^2)^3}}$  **12.** $\dfrac{a^2}{\sqrt{(a^2+x^2)^3}}$  **13.** $(1+x)^2(1-x)(1-5x)$

**14.** $-10x(1-x)(2x+3)^2$  **15.** $(x+a)^{p-1}(x+b)^{q-1}\{(p+q)x+bp+aq\}$

**16.** $-\dfrac{a^2}{x^2\sqrt{(a^2+x^2)}}$  **17.** $-\dfrac{a^2}{x^2\sqrt{(a^2-x^2)}}$  **18.** $\dfrac{-2a^2x}{(a^2+x^2)\sqrt{(a^4-x^4)}}$

**19.** $\dfrac{2na(a+x)^{n-1}}{(a-x)^{n+1}}$  **20.** $\dfrac{a(a-x)}{\sqrt{(a^2+x^2)^3}}$  **21.** $\dfrac{x^3+2a^2x-a^3}{\sqrt{(x^2+a^2)^3}}$

**22.** $n\{(1+x)^{n-1}-(1-x)^{n-1}\}$  **23.** $n^2x^{n-1}\{(1+x^n)^{n-1}-(1-x^n)^{n-1}\}$

**24.** $-ax^{-2}-a^2x^{-2}(a^2-x^2)^{-\frac{1}{2}}$  **25.** $2n(b+cx)(a+2bx+cx^2)^{n-1}$

**26.** $28x(3x+7)^{\frac{1}{3}}$  **27.** $\dfrac{15}{x^6\sqrt{(x^2-1)}}$  **28.** $\dfrac{-1+x^2-2x^4}{x^2\sqrt{(1-x^2)}}$

**29.** $21x^2\sqrt{(5+2x)}$  **30.** $-63(3-5x^2)^{\frac{3}{2}}x^8$  **31.** $\sin 2x$  **32.** $2\sin 4x$

**33.** $\frac{1}{4}\cos\frac{1}{2}x\sqrt{\operatorname{cosec}\frac{1}{2}x}$  **34.** $-2\sin 2x$  **35.** $-\sin 2x\sqrt{\sec 2x}$

**36.** $4\sec^2 2x\tan 2x$  **37.** $0.$  **38.** $2\sin 2x$  **39.** $-3\sin 3x$

**40.** $9\tan^2 3x\sec^2 3x$  **41.** $3\sec^4 x$  **42.** $3\sec x\tan^3 x$

**43.** $-2\cos x(x\sin x+\cos x)/x^3$  **44.** $2x(\sin x-x\cos x)\operatorname{cosec}^3 x$

**45.** $4\tan 2(x+a)\sec^2 2(x+a)$  **46.** $\cos x(5-4\cos^2 x)(4\cos^2 x-3)^{-2}$

**47.** $-(1+\sin x)^{-1}$  **48.** $-\cos x(4+\operatorname{cosec}^2 x)$  **49.** $\sin^2 x$  **50.** $\sin^3 x$

51 $\cos^4 x$ 52. $-\cot^2 x$ 53. $\tan^2 x$ 54. $\tan^4 x$

55. $\frac{5}{3}x^2, px^{p-q}/q$ 56. $\dfrac{ad-bc}{a'd'-b'c'}\,\dfrac{(c'x+d')^2}{(cx+d)^2}$ 57. $-1$

58. $-\cot x,\ -3\cos 3x/2\sin 2x$ 59. $-\frac{2}{3}\sec x$ 60. $\operatorname{cosec} x, \sin x$

## Exercises VII. P. 89

1. i. $2y-x=3$ ii. $3x+2y+12=0$ iii. $4x-y=5$ iv. $3y-4x=5$
v. $82x-39y=168$ vi. $3x+y=5$ vii. $13x-3y=27$
viii. $2y-\sqrt{3}x=1-\frac{1}{6}\sqrt{3}\pi$ ix. $4x-y=\pi-1$ x. $x+\pi y=\pi$

4. $x/a+y/b=2$; $x=ka^{\frac{n}{n-2}}$, $y=kb^{\frac{n}{n-2}}$, $k^n\left[a^{\frac{2n}{n-2}}+b^{\frac{2n}{n-2}}\right]=2$; $n$ even, $>2$, four normals, $n$ odd, one.

5. Tangents, $y-2x=a$, $y=2a$; normals, $x+2y=2a$, $x=a$

8. $x=2, \frac{5}{3}$ 18. $2xt-y(1-t^2)=at^2$; $y=0, y=a$

## Exercises IX. P. 119

(*A constant* C *may be added to each answer*)

1. $x+\frac{1}{2}x^2+\frac{1}{3}x^3$ 2. $\frac{1}{3}x^3-\frac{1}{2}x^2-2x$ 3. $-\frac{1}{4}(1-x)^4$

4. $x+x^{-1}+\frac{1}{2}x^{-2}$ 5. $a^4x-\frac{2}{3}a^2x^3+\frac{1}{5}x^5$ 6. $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$

7. $\frac{1}{2}ax^2+\frac{1}{4}bx^4$ 8. $\frac{1}{3}acx^3+\frac{1}{2}(ad+bc)x^2+bdx$ 9. $x^{p+1}/(p+1)+x^{1-p}/(1-p)$

10. $\frac{1}{3}x^3+2\sqrt{x}$ 11. $2\sqrt{x}(\frac{1}{3}x+1)$ 12. $-x^{-1}-\frac{1}{2}x^{-2}-\frac{1}{3}x^{-3}$

13. $\frac{1}{3}x^3-\log x$ 14. $\frac{1}{3}x^3+\frac{1}{2}x^2+x+\log(x-1)$ 15. $-\frac{1}{2}\log(1-2x)$

16. $x-2\log(2x+3)$ 17. $\log[(x-2)^3/(x-1)^2]$ 18. $\frac{1}{4}\log[(2+x)/(2-x)]$

19. $\frac{1}{2}\log[(x-3)/(x-1)]$ 20. $x+\log(x^2-x+1)$

21. $\frac{1}{4}a^{-2}\log[(x^2-a^2)/(x^2+a^2)]$ 22. $-\frac{1}{2}(x^2+a^2)^{-1}$

23. $\frac{1}{3}x^3-\frac{1}{2}x^2+x-2\log(x+1)$ 24. $\log[(2x+1)^{\frac{1}{2}}(x-1)]$

25. $\log[(3x-2)/(2x+3)]$ 26. $\log[x^{-1}(x-1)^2/(x+1)]$

27. $\log[x^{-1}(x+4)]-4/x$ 28. $\frac{3}{4}x^{\frac{4}{3}}$ 29. $-\frac{1}{2}x^{-2}$

30. $\frac{1}{3}\sqrt{(2x+1)^3}$ 31. $-\frac{1}{2}(2x+1)^{-1}$ 32. $(ax+b)^{n+1}/(an+a)$

33. $\frac{1}{3}x^3-2x-x^{-1}$ 34. $\frac{1}{5}x^5+\frac{1}{2}x^4+x^3+x^2+x$ 35. $x^3+x^2-x$

36. $\frac{1}{2}x^2+2x+\log x$ 37. $\frac{1}{3}x^3-\frac{1}{2}x^2+x-\log(x+1)$ 38. $\frac{1}{3}(x^2-3)^2/x$

39. $(\frac{2}{5}x^2-\frac{4}{3}x+2)\sqrt{x}$ 40. $\log[x^{-1}(x-1)]$ 41. $\log[x^3(x-1)^{-2}]$

42. $x+\log[x^2/(x-1)]$ 43. $\frac{1}{3}\log(x^3+3x-5)$ 44. $x-\log(x^2+x+1$

45. $\frac{1}{2}\log[(x^2-x+1)/(x^2+x+1)]$ 46. $x+2\log x-1/x$

47. $x-2\log(x+1)-(x+1)^{-1}$ 48. $5\log(2x-3)-29/(2x-3)$

49. $\frac{1}{2}\log[x(x+2)(x+1)^{-2}]$ 50. $\log x-2(2x-1)^{-1}$

51. $\frac{1}{3}\log(x^3-1)$ 52. $\frac{1}{3}\log[(x-1)^2/(x^2+x+1)]$

53. $\frac{1}{2}\log[x^{-4}(x-1)^5/(x+1)]$ 54. $\frac{1}{4}\log[(x+1)(x-1)^3]-\frac{1}{2}(x-1)^{-1}$

55. $\frac{1}{4}(x^4+4x^3+12x^2+40x)-\frac{1}{2}(12x-11)(x-1)^{-2}+15\log(x-1)$

56. $\frac{1}{3}\sqrt{(a^2+x^2)^3}$ 57. $\sqrt{(a^2+x^2)}$ 58. $(a^n+x^n)^{p+1}/(np+n)$

59. $\frac{1}{2}(\log x)^2$ 60. $\log\log x$ 61. $-\frac{1}{3}\cos 3x$

62. $\cos 9x - 9\cos 3x$ 63. $x+\sin^2 x$ 64. $\log(1+\sin x)$

65. $-\log(1-\sin x)$ 66. $\tan x - x$ 67. $\frac{1}{3}\tan^3 x - \tan x + x$

68. $\frac{1}{2}\tan^2 x + \log\cos x$ 69. $\frac{1}{3}\tan^3 x + \tan x$ 70. $\frac{1}{3}\sec^3 x - \sec x$

71. $\frac{1}{2}x+\frac{1}{2}\log(\sin x+\cos x)$ 72. $[(ac+bd)x+(bc-ad)\log(c\sin x+d\cos x)]/(c^2+d^2)$

73. $-\frac{1}{2}\frac{\cos(m+n)x}{m+n}-\frac{1}{2}\frac{\cos(m-n)x}{m-n}$ 74. $\frac{1}{2}\frac{\sin(m-n)x}{m-n}-\frac{1}{2}\frac{\sin(m+n)x}{m+n}$

75. $\frac{1}{2}\frac{\sin(m-n)x}{m-n}+\frac{1}{2}\frac{\sin(m+n)x}{m+n}$

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